Characterizations of L^p-spaces

with $p \in (0,\infty)$

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Characterizations of L^p-spaces with $p \in (0, \infty)$

een wetenschappelijke proeve op het gebied
van de Natuurwetenschappen, Wiskunde en Informatica

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door

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Je promotietijd is als je tijd in militaire dienst. Achteraf houd je alleen de goede herinneringen over, maar als je er middenin zit, word je hard met je eigen onvermogens geconfronteerd.

Vrij naar Robbert Dijkgraaf,
hoogleraar Mathematische Fysica
(Volkskrant Magazine, "Beroep: student", 1998)

De worsteling met problemen die "groter zijn dan jezelf" heeft significant bijgedragen aan mijn ontwikkeling, in wetenschappelijk en persoonlijk opzicht.
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Steven Teerenstra

Introduction and Summary

Let S be a set, $\mathcal{A}$ a σ-algebra in S, and μ a measure on $\mathcal{A}$.
For any $p \in (0, \infty)$, one can define

$$L^p(\mu): \left\{ \begin{array}{l} \textit{the collection of all real } \mathcal{A}\textit{-measurable functions } f \textit{ on } S \textit{ for which} \\ \qquad \|f\|_p := [\int_S |f|^p d\mu]^{1/p} < \infty \\ \textit{with identification of functions that are equal } \mu\textit{-almost everywhere.} \end{array} \right.$$

In 1940, Kakutani and Bohnenblust gave a characterization of the spaces $L^p(\mu)$ with $p \in [1, \infty)$. In their characterization they not only took into account the linear and metric structure of L^p-spaces, but also their natural ordering structure:

$$f \le g \quad \Longleftrightarrow \quad f(s) \le g(s) \textit{ for } \mu\textit{-almost all } s \in S\ .$$

Roughly stated, they found that the property that sets aside L^p-spaces from other spaces with similar linear, metric and ordering structure, is *p-additivity*:

$$\|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p \quad \textit{whenever } f \textit{ and } g \textit{ are disjoint.}$$

The latter is a translation of the fact that real functions f, g with *disjoint* support (i.e. $|f(s)| > 0 \Longrightarrow |g(s)| = 0$ for all s) have the property that

$$|f + g|^p(s) = |f|^p(s) + |g|^p(s) \ (p \in (0, \infty)\ , \quad s \in S)\ .$$

p-additiviteit beperkt zich echter niet alleen tot $p \in [1, \infty)$. Het is dus à priori geen hopeloze zaak om te proberen de Kakutani-Bohnenblust karakterisering te generaliseren zó dat ook L^p-ruimtes met $p \in (0, 1)$ eronder vallen.
In hoofdstuk 1 wordt zo'n generalisatie bewezen; daarnaast worden twee andere karakteriseringen van L^p-ruimtes ($p \in (0, \infty)$) gepresenteerd. Alle drie deze karakteriseringen richten zich op de lineaire, ordenings- en metrische structuur.

Almost simultaneously with their characterization of L^p in terms of p-additivity with $p \in [1, \infty)$, Kakutani and Bohnenblust obtained a characterization concerning

$$C(S): \left\{ \begin{array}{l} \textit{the collection of all real continuous functions } f \textit{ on } S \textit{ for which} \\ \qquad \|f\|_\infty := \sup\{|f(s)| : s \in S\} < \infty, \end{array} \right.$$

where S is a compact Hausdorff space. To be precise, they showed that linear subspaces of $C(S)$ that are closed with respect to $\|\ \|_\infty$ and closed with respect to taking pointwise maxima and minima (so called *M-spaces*), can be characterized by *∞-additivity*:

$$\|f + g\|_\infty = \max\{\|f\|_\infty, \|g\|_\infty\} \quad \textit{if } f \textit{ and } g \textit{ are disjoint.}$$

Thus, L^p-spaces with $p \in [1, \infty)$ and M-spaces can be jointly characterized by p-additivity ($p \in [1, \infty]$). One can even dispose of specifying the parameter p:

Bohnenblust showed that, in the context of vector spaces with similar ordering and metric structure, every norm $\|\ \|$ with the following homogeneity property

$$\left.\begin{array}{l} f_1, g_1 \textit{ disjoint}, \\ f_2, g_2 \textit{ disjoint}, \\ \|f_1\| = \|f_2\|, \\ \|g_1\| = \|g_2\|, \end{array}\right\} \implies \|f_1 + g_1\| = \|f_2 + g_2\| \qquad (*)$$

is necessarily p-additive for some $p \in [1, \infty]$.
Based on the above Kakutani-Bohnenblust results, others obtained joint characterizations of L^p-spaces with $p \in [1, \infty)$ and (subclasses of) M-spaces, of which we will consider:

1. Ando's characterization in terms of positive contractive projections, and
2. one of Tzafriri's characterizations in terms of convergence behavior of infinite disjoint sequences.

Ando's characterization does not allow a generalization that includes L^p-spaces with $p \in (0, 1)$, the argument for which is provided by a characterization of contractive projections in L^p, also due to Ando. The end of chapter 1 presents an alternative proof of Ando's characterization of contractive projections in L^p for the case that $p \in (0, 1)$. It is more elementary and yields a slightly more general result.
Chapter 2 is devoted to generalizing one of Tzafriri's characterizations of L^p-spaces and M-spaces. Like the characterizations mentioned earlier, Tzafriri's characterization uses the linear and order structure, but in contrast with them, the *topological* structure is exploited instead of the metric structure. In Tzafriri's characterization, the p-additivity is replaced by (or better termed, hidden in) the condition that all disjoint normalized infinite sequences are equivalent:

If $u_1, u_2, \ldots$ disjoint, $v_1, v_2, \ldots$ disjoint, and $\|u_i\| = \|v_i\|$ for all i,
then for every sequence of scalars $\lambda_1, \lambda_2, \ldots$

$$\textstyle\sum_i \lambda_i u_i \textit{ converges} \iff \sum_i \lambda_i v_i \textit{ converges},$$

which is a sort of infinite dimensional analogue of $(*)$.
Since L^p-spaces do not fall under the regime of normed spaces for $p \in (0, 1)$, a generalization of Tzafriri's characterization that is to include L^p-spaces with $p \in (0, 1)$, must be put in a more general setting: that of quasi-normed spaces. This leads to a joint characterization of L^p-spaces and, what we will call, quasi-M-spaces. M-spaces are trivially quasi-M-spaces, but the converse question remains open. Chapter 2 ends with a discussion of some sufficient conditions on a quasi-M-space to be an M-space.

Guide to the reader

A PhD-thesis always ends up somewhere between a textbook and a bunch of articles. In my case, the result looks more like the former, not in the last place, because I had two audiences in mind:

- experts in Riesz space theory, and
- (other) interested readers who are not afraid of functional analysis.

Also, my intention to make the thesis reasonably self-contained and accessible to the latter audience, blew up the preliminaries from 20 pages to 40 pages before I knew it. I can only ask the reader's indulgence and give my advice about what is worth reading.

To experts:
The main results are presented in the sections 1.1, 1.3, and 2.4. The first two of those concern Riesz isometric characterizations of L^p-spaces, and the last deals with Riesz homeomorphic characterizations. Side-lines of the main theme can be found in

- 1.2: characterizations of dense Riesz subspace of L^p;
- 1.4: an alternative proof of Ando's characterizations of contractive projections in L^p, $p \in (0,1)$;
- 1.5: characterizations of band projections among the contractive projections in L^p, $p \in (0,1)$;
- 2.5: the question whether quasi-M-spaces are M-spaces.

The beginning sections of chapter 2 (2.1-2.3) as well as the preliminaries can be considered as a "reference manual" which can be consulted whenever one feels the need. The index has been prepared with this goal in mind. As a rule, the parts of text that are typeset in a small typeface are meant as elucidations or elaborations for non-experts, and can be omitted. Results of preliminaries that play a crucial rôle in chapter 1 or chapter 2, are referenced when used. Most sections end with remarks which are primarily meant for you.

To other interested readers:
I presuppose that you have some knowledge of topology and functional analysis. To give an idea: concepts and theorems that will be used without definition or proof, are:

- from topology: Hausdorff space, Urysohn's lemma, compactness;
- from measure theory: σ-finite measure, the construction of the integral and of L^1, almost everywhere, Levi's monotone convergence theorem;

- from topological vector space theory: normed space, Banach space, topological vector space, the uniform boundedness principle and its derivatives: the open mapping theorem and the closed graph theorem.

If desired, lacunas in this respect can be filled in by consulting [Kö] or [Ed] for instance.
From the above starting point, the preliminaries seriously try to get the interested reader up to speed concerning the relevant theory of L^p-spaces, Riesz spaces, quasi-normed (Riesz) spaces, and, to a lesser extent, locally solid Riesz spaces. After that, most sections of chapter 1 and 2 (without their ending remarks), are hopefully accessible. Since I expect that the sections 1.1 and 2.4 are more interesting to you than the rest, I have marked the parts of the preliminaries that are *not* needed for understanding 1.1 and 2.4 with a #. In particular, the sections 0.3.6, 0.3.7, 0.3.9, 0.3.12, 0.3.13, 0.6.1, and 0.6.2, may be omitted. The sections 2.1-2.3 build up some theory that is needed to state and prove the results in 2.4. As a rule, the parts of text that are typeset in a smaller type face are meant as stepping stones which elaborate some details of the line of argument.

Finally, this place is perhaps as good as any other to mention that we will assume the Axiom of Choice and will work with Archimedean Riesz spaces only.

Notation

$\mathbb{N}$ is the set of positive integers;
$\mathbb{N}_0$ is the set of non-negative integers;
$\mathbb{R}$ is the set of real numbers;
$\mathbb{Q}$ is the set of rational numbers;
c is the vector space of convergent sequences.

Let S be a set.
If for some $s \in S$ a statement $P(s)$ is defined, then

$$[\, P \,] := \{s \in S : P(s) \text{ is true }\}$$

is the set of all $s \in S$ for which $P(s)$ is (defined and) true.
E.g. if $f : S \to \mathbb{R}$, then $[f > 0] = \{s \in S : f(s) > 0\}$.

Some vector spaces (normed spaces) of functions on S:

$\ell^\infty(S)$: the vector space of all functions $f : S \to \mathbb{R}$ for which $\| f \|_\infty := \sup\{|f(s)| : s \in S\} < \infty$;
$c_0(S)$: the vector space of all functions $f : S \to \mathbb{R}$ for which the sets $[|f| > \varepsilon]$, $\varepsilon \in (0, \infty)$, are finite;
$c_{00}(S)$: the vector space of all functions $f : S \to \mathbb{R}$ with finite support, i.e. for which the set $[|f| > 0]$ is finite,

and some functions:

$\varkappa$ is the identity function $S \to S$, $s \mapsto s$;
$\mathbb{1}_A$ is the indicator function of $A \subset S$, $\mathbb{1}_A(s) = 1$ on A and $\mathbb{1}_A(s) = 0$ elsewhere.

If S is a topological space:

U° is the interior of a subset U of S;
$\overline{U}$ is the closure of a subset U of S;
Borel(S) is the σ-algebra of Borel sets of S;
βS is the Stone-Čech compactification of S;
BC(S) is the vector space of bounded continuous functions on S.

If S is a compact Hausdorff space:

$C(S)$ is the collection of continuous functions on S;
$C^\infty(S)$ is the collection of extended real-valued functions on S (see p. 7);

Miscellaneous:

$\sum_1^n x_\iota = x$ disjoint	:	$x_1, \ldots, x_n$ are (mutually) disjoint and have x as sum (p. 14);
$E_{[e]}$	:	the principal ideal of E generated by e (p. 11);
E^δ	:	the Dedekind completion of E (p. 22);
$\mathcal{L}^p_{\mathbb{C}}$	:	the complex-valued functions on S whose p^{th} powers are integrable ;
$\rfloor\!\rfloor\ \lfloor\!\lfloor$		will generally denote a quasi-norm (see p. 29);
$\Vert\ \Vert$		will denote a norm in general;
$[-\infty, \infty]$	:	the extended real number system: $\mathbb{R} \cup \{\pm\infty\}$;
$[[e_1, e_2, \ldots]]$	:	the linear span of $\{e_1, e_2, \ldots\}$ i.e. $\{\sum_1^n \lambda(i) e_{\alpha(\iota)} : n \in \mathbb{N},\ \alpha(1) < \ldots < \alpha(n),\ \lambda(i) \in \mathbb{R}\}$;
$\mathrm{Re}(f)$	:	the real part of a function f;
$\mathrm{Im}(f)$	:	the imaginary part of a function f;
$:=$	:	"is defined (by equality) as" ;
$:\Longleftrightarrow$	:	"is defined (by equivalence) as";
$x \underset{K}{\sim} y$	:	$K^{-1} x(n) \le y(n) \le K x(n)$ for all n (see 2.18 on p. 77);
$1 \wedge p,\ 1 \vee p$	:	$\min\{1, p\}, \max\{1, p\}$ in $\mathbb{R}$;
$u_\alpha \uparrow (u)$	:	(u_α) is upwards directed (to u) (see p. 5);
$u_\alpha \downarrow (u)$	:	(u_α) is downwards directed (to u) (see p. 5).

We often write things like "$0 \le f_n \uparrow\le f$ in E". In such cases, the phrase "in E" refers to the fact that the ordering of E is meant and that all elements mentioned are in E.

Finally, if $(x_n)_n$ is a sequence, then we denote the sequence of partial sums (the series) by

$$\textstyle\sum_n x_n \quad \text{i.e.} \quad \sum_n x_n = \left(\sum_1^N x_n\right)_{N=1}^{\infty}.$$

If the series $\sum_n x_n$ converges, then we denote its limit by $\sum_1^\infty x_n$. For example,

$$\textstyle\sum_n x_n \to a \quad \text{means that} \quad \sum_1^\infty x_n = a.$$

Preliminaries

In chapter 1 and 2, generalizations of characterizations of L^p-spaces will be studied. Apart from the linear structure, two other structures of interest will be used there to characterize L^p-spaces:

1. the metric structure, which is that of a *quasi-normed space*, and
2. the order structure, which is that of a *Riesz space*.

In the next four sections we will discuss the relevant theory concerning

- the order structure of a Riesz space,
- the metric structure of a quasi-normed space,
- the natural interaction between the two structures above as implemented by a quasi-normed Riesz space,
- and a extension thereof that is comprised in the concept of a locally solid Riesz space.

We begin, however, with defining L^p-spaces and collecting some of their properties.

0.1 L^p-spaces

Let $(S, \mathcal{A}, \mu)$ be a measure space, and let $p \in (0, \infty)$. Denote by $\mathcal{L}^p(S, \mathcal{A}, \mu)$, or $\mathcal{L}^p(\mu)$ or $\mathcal{L}^p$ for short, the collection of all $\mathcal{A}$-measurable functions $f : S \to \mathbb{R}$ with $||f||_p := [\int_S |f|^p \, d\mu]^{1/p} < \infty$. Since for functions $f, g : S \to \mathbb{R}$

$$\left.\begin{array}{l} |f+g|^p \\ |\max\{f,g\}|^p \\ |\min\{f,g\}|^p \end{array}\right\} \leq 2^p \max\{|f|^p, |g|^p\} \leq 2^p(|f|^p + |g|^p) \quad \text{pointwise on } S\,,$$

we have that

1. the pointwise sum of two functions in $\mathcal{L}^p$ is in $\mathcal{L}^p$, as well as
2. their pointwise maximum and minimum.

Identifying functions in $\mathcal{L}^p(\mu)$ that are equal μ-almost everywhere we obtain $L^p(\mu)$: the collection of all equivalence classes

$$[f]_\mu := \{h \in \mathcal{L}^p : h(s) = f(s) \text{ for } \mu\text{-almost all } s \in S\}\,,\ f \in \mathcal{L}^p(\mu)\,.$$

By 1. (and standard measure theoretic arguments) the following operations, function, and relation, respectively, are well-defined:

$$\begin{aligned} [f]_\mu + [g]_\mu &:= \{h \in \mathcal{L}^p : h(s) = f(s) + g(s) \text{ for } \mu\text{-almost all } s \in S\}\,, \\ r[f]_\mu &:= \{h \in \mathcal{L}^p : h(s) = rf(s) \text{ for } \mu\text{-almost all } s \in S\}\,, \\ ||\,[f]_\mu||_p &:= [\textstyle\int_S |h|^p]^{1/p} \quad (h \in [f]_\mu)\,, \\ [f]_\mu \leq [g]_\mu &:\Longleftrightarrow f(s) \leq g(s) \text{ for } \mu\text{-almost all } s \in S\,, \end{aligned}$$

In the special case that the measure space $(S, \mathcal{A}, \mu)$ consists of the set of natural numbers $\mathbb{N}$ equipped with the σ-algebra of all subsets of $\mathbb{N}$ and the counting measure, we write ℓ^p for $L^p(\mu)$. Observe that $[x]_\mu$ coincides with $\{x\}$ in that case. From now on we will, as usual, suppress the distinction between the equivalence class $[f]$ and a(ny) representative of it.

We collect some properties of L^p:

- By 1., $(L^p, +, \cdot)$ is a real vector space.
- By 2., $(L^p, \leq)$ is a lattice: a partially ordered set such that every doubleton $\{f, g\}$ has a least upper bound $f \vee g$ and greatest lower bound $f \wedge g$ (in this case given by $(f \vee g)(s) = \max\{f(s), g(s)\}$ and $(f \wedge g)(s) = \min\{f(s), g(s)\}$ for μ-almost all $s \in S$).
- $(L^p, +, \cdot, \leq)$ is a partially ordered vector space i.e. a real vector space endowed with an ordering that is compatible with the linear structure in the following sense: for all elements f, g, h
$$\begin{aligned} f \leq g \quad &\Rightarrow \quad f + h \leq g + h\,, \\ r \in [0, \infty)\,,\ 0 \leq f \quad &\Rightarrow \quad 0 \leq rf\,. \end{aligned}$$
- The functional $\|\ \|_p$ is Riesz, meaning that for elements $f, g \in L^p$
$$|f| \leq |g| \quad \Rightarrow \quad \|f\|_p \leq \|g\|_p\,.$$
- p-additivity of $\|\ \|_p$:
$$f \wedge g = 0 \text{ in } L^p(\mu) \quad \Rightarrow \quad \|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p \qquad (f, g \in L^p(\mu))\,;$$

$\lhd f \wedge g = 0$ in $L^p(\mu)$ implies that $\min\{f(s)^p, g(s)^p\} = 0$ for μ-almost all $s \in S$ (in particular, $f, g \geq 0$ μ-almost everywhere), and thus $[f(s) + g(s)]^p(s) = f^p(s) + g^p(s)$ for almost all $s \in S$.$\rhd$

- $\|\ \|_p$-inequalities

	$0 \leq u, v \in L^p(\mu)$	$f, g \in L^p(\mu)$
$0 < p \leq 1$	$\|u\|_p + \|v\|_p \leq \|u + v\|_p$	$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$
$1 \leq p < \infty$	$\|u\|_p^p + \|v\|_p^p \leq \|u + v\|_p^p$	$\|f + g\|_p \leq \|f\|_p + \|g\|_p$

Proof

(I): The inequalities in the upper left corner and lower right corner

Let $0 \leq u, v \in L^p(\mu)$. Leaving aside the trivial case that either $u = 0$ or $v = 0$, we consider $\|u + v\|_p$ for $p \in (0, 1]$ and $p \in [1, \infty)$ respectively.

Case $0 < p \leq 1$

The function $t \mapsto t^p$, $[0, \infty) \to [0, \infty)$ is concave, i.e.
$$\left.\begin{array}{l} \alpha, \beta \in [0, 1]\,,\ \alpha + \beta = 1 \\ t, t' \in [0, \infty) \end{array}\right\} \Rightarrow (\alpha t + \beta t')^p \geq \alpha t^p + \beta (t')^p\,. \tag{\&}$$

If we apply the above concavity property (for each $s \in S$) with
$$\alpha := \frac{\|u\|_p}{\|u\|_p + \|v\|_p}\,,\ \beta := \frac{\|v\|_p}{\|u\|_p + \|v\|_p}\,,\ t := \frac{u(s)}{\|u\|_p}\,,\ t' := \frac{v(s)}{\|v\|_p}\,,$$
we see that
$$\begin{aligned} \left\|\frac{u + v}{\|u\|_p + \|v\|_p}\right\|_p^p &= \int_S \left[\alpha \frac{u(s)}{\|u\|_p} + \beta \frac{v(s)}{\|v\|_p}\right]^p d\mu(s) \\ &\geq \int_S \left[\alpha \left[\frac{u(s)}{\|u\|_p}\right]^p + \beta \left[\frac{v(s)}{\|v\|_p}\right]^p\right] d\mu(s) = \alpha \cdot \frac{\int |u|^p}{\|u\|_p^p} + \beta \cdot \frac{\int |v|^p}{\|v\|_p^p} = 1\,, \end{aligned} \tag{\%}$$

which establishes that $\|u+v\|_p \geq \|u\|_p + \|v\|_p$ for $0 \leq u, v \in L^p(\mu)$.

Case $1 \leq p < \infty$
The function $t \mapsto t^p$, $[0,\infty) \to [0,\infty)$ is now convex, i.e. the inequality in (&) reverses, which implies in its turn that the inequality in (%) reverses. Thus, $\|u+v\|_p \leq \|u\|_p + \|v\|_p$ for all $0 \leq u, v \in L^p(\mu)$.

As a consequence, we have for all $f, g \in L^p(\mu)$:

$$\|f+g\|_p = \| \, |f+g| \, \|_p \leq \| \, |f|+|g| \, \|_p \leq \| \, |f| \, \|_p + \| \, |g| \, \|_p = \|f\|_p + \|g\|_p .$$

(II): The inequalities in the lower left corner and upper right corner
We consider $\| u+v \|_p^p$ for $p \in (0,1]$ and $p \in [1,\infty)$ respectively.

Case $0 < p \leq 1$
Since $t^p \geq t$ $(t \in [0,1])$ for $p \in (0,1)$, we have that $(\frac{r}{r+s})^p + (\frac{s}{r+s})^p \geq 1$ for $r, s \in (0,\infty]$. In other words,

$$(r+s)^p \leq r^p + s^p \quad (r, s \in [0,\infty)) . \tag{⋔}$$

Applying (⋔) pointwise to $0 \leq u, v$ in L^p, yields $\| u+v \|_p^p \leq \| u \|_p^p + \| v \|_p^p$. By a similar argument as in *(I)*, the latter inequality extends to $f, g \in L^p$.

Case $1 \leq p < \infty$
For this kind of p, $t^p \leq t$ $(t \in [0,1])$, and so the inequality in (⋔) reverses, which implies that $\| u+v \|_p^p \geq \| u \|_p^p + \| v \|_p^p$. □

- From the $\| \, \|_p$-inequalities above it is clear that $\| \, \|_p$ is a *norm* for $p \in [1,\infty)$. *However, for $p \in (0,1]$, $\| \, \|_p$ is no norm:* if μ is the Lebesgue measure on $(\mathbb{R}, \mathrm{Borel}(\mathbb{R}))$, then

$$\|\mathbb{1}_{[0,2]}\|_p = 2^{1/p} \nleq 2 = \|\mathbb{1}_{[0,1]}\|_p + \|\mathbb{1}_{[1,2]}\|_p \; . \quad (\| \, \|_p \text{ not subadditive for } p \in (0,1))$$

However, we do have a weakened form of subadditivity:

$$\|f+g\|_p \leq 2^{(1-p)/p}(\|f\|_p + \|g\|_p) \quad (p \in (0,1] , \; f, g \in L^p(\mu)) . \tag{quasi-subadditivity}$$

For a proof of this, use that $\| f+g \|_p^p \leq \| f \|_p^p + \| g \|_p^p$ and the following

Lemma 0.1
$$(r^p + s^p)^{1/p} \leq 2^{(1-p)/p}(r+s) \quad (r, s \in [0,\infty) , \; p \in (0,1]).$$

◁The proof of lemma 0.1 relies again on the concavity of $t \mapsto t^p$ $(t \in [0,\infty))$: applying the concavity property

$$(\alpha t' + \beta t)^p \geq \alpha t'^p + \beta t^p$$

to $\alpha = \beta = \frac{1}{2}$ and $t' = \frac{r}{r+s}$, $t = \frac{s}{r+s}$ gives the desired result.▷

0.2 Lattices

Since the lattice structure of L^p will be relevant to us, we introduce some standard terminology and ditto results.

Definition 0.2

Let L be a set, let $\leq$ be a partial ordering in L, and let $f, g \in L$.

i. *We use the convention of writing $f < g$ if $f \leq g$ and $f \neq g$.*

ii. *If $(L, \leq)$ is a lattice, we denote the least upper bound and lower bound of $\{f, g\}$ by $f \vee g$ and $f \wedge g$, respectively. The operations $\vee$, $\wedge$ are referred to as the lattice operations.*

iii. *$L_0 \subset L$ is called a sublattice if $f, g \in L_0$ implies that $f \vee g, f \wedge g \in L_0$*

iv. *A subset $U \subset L$ is called upwards directed (downwards directed) if for all u, v in U there exists a $w \in U$ such that $u \leq w$ and $v \leq w$ ($w \leq u$ and $w \leq v$, respectively). With $U \uparrow a$ ($U \downarrow a$) we denote that U is upwards directed and $\sup U = a$ (respectively: U is downwards directed and $\inf U = a$).*

   We say that a net $(u_\alpha)_\alpha$ in L is upwards directed (increasing), notation $u_\alpha \uparrow$, if $u_{\alpha_1} \leq u_{\alpha_2}$ in L, whenever $\alpha_1 \prec \alpha_2$. If, in addition, $\sup_\alpha u_\alpha = u$ for some $u \in L$, we say that $(u_\alpha)_\alpha$ is upwards directed to u, and we denote this with $u_\alpha \uparrow u$. The meaning of "$(u_\alpha)_\alpha$ is downwards directed (to u)", notation $u_\alpha \downarrow$ ($u_\alpha \downarrow u$), is analogously defined.

v. *A subset $U \subset L$ is called order bounded from above (from below) if there exists a $v \in L$ such that for all $u \in U$: $u \leq v$ ($u \geq v$ respectively). Instead of "order bounded from above" the term majorized is also used. When U is both order bounded from above and from below, we shortly say that U is order bounded.*

vi. *L is called (σ)-Dedekind complete, whenever every (countable) non-empty, order bounded set in L has a supremum and an infimum in L.*

Examples 0.3

- $f < g$ in $L^p(\mu)$ means that for almost all $s \in S$: $f(s) \leq g(s)$, and that the set $\{s \in S : f(s) < g(s)\}$ is non-negligible.
- $L^p(\mu)$ is σ-Dedekind complete:
  if $\{f_n\}_n$ is order bounded, then $(L^p\text{-}\sup_n f_n)(s) = \sup_n f_n(s)$ for μ-almost all $s \in S$, and $(L^p\text{-}\inf_n f_n)(s) = \inf_n f_n(s)$ for μ-almost all $s \in S$. In particular, if $f_n \downarrow 0$ in L^p, then $\inf_n f_n(s) = 0$ for μ-almost all $s \in S$.
- $\ell^\infty(S)$ is Dedekind complete.
- $\{x \in \ell^\infty(S) : x$ is constant outside a countable set$\}$ is σ-Dedekind complete, but not Dedekind complete if S is uncountable.
- $\{x \in \ell^p : x$ assumes values in a finite set$\}$ is not σ-Dedekind complete, and neither is $C[0,1]$.

In fact, L^p is Dedekind complete as we will see in 0.14. As a preparation we show:

Lemma 0.4 (super-Levi property of L^p-spaces)

Let $p \in (0, \infty)$, and let $F \subset \{g \in L^p : g \geq 0\}$ be an upwards directed set such that $M := \sup\{\|f\|_p : f \in F\} < \infty$.

Then $\sup F$ exists in L^p, and there exists a countable subset $\{f_n\}_n$ of F such that $\sup_n f_n = \sup F$.

◁By Levi's monotone convergence theorem every increasing sequence $(g_n)_n$ in F has a supremum g_∞ and $||g_\infty||_p \le M$. Let F_∞ be the collection of those suprema. Observe that $F_\infty \supset F$, that the upper bounds of F_∞ and F coincide, and that a countable subset of F_∞ corresponds to one of F. We prove that there exists a subset $\{f_n\}_n$ of F_∞ such that $\sup_n f_n = \sup F_\infty (= \sup F)$.
Indeed, select $0 \le f_n \uparrow$ with $||f_n||_p \ge M - n^{-1}$. Then $f_\infty = \sup_n f_n \in F_\infty$ and $||f_\infty||_p = M$. We show that $f_\infty = \max F_\infty = \sup F_\infty$. To this end, let $f \in F_\infty$. Then $f \vee f_\infty \in F_\infty$ (hence $||f \vee f_\infty||_p \le M$), and using a $|| \; ||_p$-inequality

$$||f \vee f_\infty - f_\infty||_p^{1 \vee p} \le ||f \vee f_\infty||_p^{1 \vee p} - ||f_\infty||_p^{1 \vee p} \le M^{1 \vee p} - ||f_\infty||_p^{1 \vee p} = 0 \,,$$

we see that $f_\infty = f \vee f_\infty \ge f$.▷

0.3 Riesz spaces

In this section we discuss the symbiosis of the lattice structure and the linear structure in an L^p-space in terms of its Riesz space structure.

Definition 0.5

A Riesz space is a real linear space E that is partially ordered such that $(E, +, \cdot, \le)$ is a partially ordered vector space and $(E, \le)$ is a lattice.
Let E be a Riesz space. Then $f \in E$ is called positive if $f \ge 0$, and the collection of all positive elements in E, denoted by E^+, is called the positive cone.
For $f \in E$, the positive part, negative part, and absolute value of f are defined by $f^+ := f \vee 0$, $f^- := (-f) \vee 0$ and $|f| := f \vee (-f)$.
The operations $+, \vee, \wedge : E \times E \to E$; $^+, ^- : E \to E$ and the scalar multiplication $\cdot : \mathbb{R} \times E \to E$ are referred to as the Riesz space operations.

From section 0.1 above, it follows that L^p-spaces are Riesz spaces.

0.3.1 Generic examples of Riesz spaces

The Riesz space $C(S)$

Let S be a compact Hausdorff space. By $C(S)$ we denote the real vector space of all continuous functions on S. Equipped with the pointwise ordering, i.e. $f \le g$ in $C(S)$ if and only if $f(s) \le g(s)$ for all $s \in S$, $C(S)$ becomes a Riesz space, and for $s \in S$

$$(f \vee g)(s) = \max\{f(s), g(s)\}\,, \qquad (f \wedge g)(s) = \min\{f(s), g(s)\}\,,$$
$$(f^+)(s) = \max\{f(s), 0\}\,, \quad (f^-)(s) = \max\{-f(s), 0\}\,, \quad |f|(s) = |f(s)|\,.$$

Simple examples of Riesz spaces are obtained by taking S finite: $\mathbb{R}^n$ with the coordinatewise ordering. In particular, $\mathbb{R}$ itself is the simplest example of a Riesz space.
A special class of functionals on $C(S)$ is formed by the point evaluations:

Definition 0.6

For $s \in S$, the functional $\delta_s : C(S) \to \mathbb{R}$ mapping $f \in C(S)$ to its value at s, $f(s)$, is called the point evaluation at s.

The following characterization of point evaluations will be of importance to us.

Lemma 0.7
Let $\omega : C(S)^+ \to [0,\infty)$ be

1. *positively homogeneous: $\omega(rf) = r\omega(f)$ ($f \in C(S)^+, r \in [0,\infty)$),*
2. *$\wedge$-preserving: $\omega(f \wedge g) = \omega(f) \wedge \omega(g)$ ($f, g \in C(S)^+$),*
3. *$\vee$-preserving: $\omega(f \vee g) = \omega(f) \vee \omega(g)$ ($f, g \in C(S)^+$), and*
4. *$\mathbb{1}$-preserving: $\omega(\mathbb{1}) = 1$.*

Then there is a unique $s \in S$ such that $\omega = \delta_s$ on $C(S)^+$. In particular, ω can be uniquely extended to a linear functional on $C(S)$.

$\triangleleft$ *(I): ω is increasing. In particular, if $f \leq \mathbb{1}$, then $\omega(f) \leq 1$.*
Indeed, if $f \leq g$, then $\omega(f) = \omega(f \wedge g) = \omega(f) \wedge \omega(g) \leq \omega(g)$.
(II) There is an $a \in S$ such that $f(a) = 0$ implies $\omega(f) = 0$ ($f \in C(S)^+$)
Suppose the contrary. Then for all $s \in S$ there exists an $f_s \in C(S)^+$ such that $f_s(s) = 0$, but $\omega(f_s) > 0$, say $\omega(f_s) > 1$. By a compactness argument, there exist $s_1, \ldots, s_n \in S$ with $S \subset \cup_1^n [f_{s_i} < 1]$. Let $f := f_{s_1} \wedge \ldots \wedge f_{s_n}$. Then $\omega(f) = \omega(f_{s_1}) \wedge \ldots \wedge \omega(f_{s_n}) > 1$, while $f \leq \mathbb{1}$, so that $\omega(f) \leq 1$. *Contradiction.*
(III): For all $f \in C(S)^+$: $\omega(f) = f(a)$.
Let $f \in C(S)^+$. The case $f(a) = 0$ follows from *(II)*, so we assume that $f(a) > 0$. First let $r \in (f(a), \infty)$ and set

$$e := \frac{[f - f(a)\mathbb{1}]^+}{r - f(a)} .$$

Then $e \in C(S)^+$, $e(a) = 0$ (so $\omega(e) = 0$), and $f \leq \|f\|_\infty e \vee r\mathbb{1}$ (distinguish the cases $s \in [f \geq r]$ and $s \in [f < r]$). Thus, $\omega(f) \leq \|f\|_\infty \omega(e) \vee r\omega(\mathbb{1}) = 0 \vee r$. The latter holds for all $r \in (f(a), \infty)$, so $\omega(f) \leq f(a)$.
Secondly, let $r \in (0, f(a))$ and set

$$e := \frac{[f(a)\mathbb{1} - f]^+}{f(a) - r}$$

Then $e \in C(S)^+$, $e(a) = 0$, and $r\mathbb{1} \leq f \vee re$ ($e \geq 1$ on $[f < r]$). ω preserves the latter inequalities, and it follows that $\omega(f) \geq f(a)$. $\triangleright$

The Riesz space $C^\infty(S)$

Let $[-\infty, \infty]$ be the extended real number system.

Definition 0.8
Let S be a topological space. An extended real-valued function on S is a continuous function from S to $[-\infty, \infty]$ such that the open set $[f \neq \pm\infty]$ is dense.
The collection of all extended real-valued functions on S is denoted by $C^\infty(S)$.

$C^\infty(S)$ is a lattice, and for $f, g \in C^\infty(S)$ and $r \in \mathbb{R}$ we have

$$\left.\begin{aligned} (f \wedge g)(s) &= \min\{f(s), g(s)\}, \\ (f \vee g)(s) &= \max\{f(s), g(s)\}, \end{aligned}\right\} \quad (s \in S).$$

A scalar multiplication and a partially defined addition in $C^\infty(S)$ are given by

$$\begin{aligned} &(rf)(s) = rf(s) \quad (s \in S, r \in \mathbb{R}), \\ &f + g = h \quad \text{if } f(s) + g(s) = h(s) \quad \text{for all } s \in [f \neq \pm\infty] \cap [g \neq \pm\infty]. \end{aligned}$$

$C^\infty(S)$ is closed under scalar multiplication, but the situation for the operation $+$ is more complicated. Some sublattices of $C^\infty(S)$ (such as $C(S)$) are closed with respect to (scalar multiplication and) addition (these are referred to as **Riesz spaces of extended real-valued functions**), and some sublattices are not.
In general, $C^\infty(S)$ itself is not closed under the above addition ([LuZa, p. 295]); this situation improves if S is sufficiently disconnected.

Definition 0.9
Let S be a compact Hausdorff space. A subset of S is called **clopen** *if it is closed as well as open. Recall that a countable union of closed subsets of S is called an* **F_σ-set.**

If the closure of every open F_σ-set is clopen, then we say that S is **basically disconnected.** *In particular, if S is basically disconnected, then for every continuous $f : S \to [-\infty, \infty]$ the closure of $[f > 0]$ is clopen.*
If in fact the closure of every open set in S is clopen, we say that S is **extremally disconnected.**

We can now formulate a sufficient condition on S for $C^\infty(S)$ to be a Riesz space:

Lemma 0.10 (cf. [LuZa, Thm. 47.1, p. 322])
Let S be a basically disconnected compact Hausdorff space.
Suppose that O is an open F_σ-set in S, and that $f : O \to \mathbb{R}$ is continuous.
Then f can be extended to a extended real-valued function $\overline{f}$ on the closure $\overline{O}$, and this extension is unique.
In particular we see that

i. *$C^\infty(S)$ is closed under the operation of vector addition, and hence $C^\infty(S)$ becomes a Riesz space;*
ii. *for every continuous $\phi : \mathbb{R} \to \mathbb{R}$ and every $f \in C^\infty(S)$ there is a unique function $g \in C^\infty(S)$ with $g(s) = \phi(f(s))$ for all $s \in [f \neq \pm\infty]$.*

Actually, the disconnectedness of S not only influences the domain of the addition in $C^\infty(S)$, but also the Dedekind completeness of the latter:

Lemma 0.11
Let S be a compact Hausdorff space. Then

i. *if S is basically disconnected, then $C^\infty(S)$ is a σ-Dedekind complete Riesz space;*
ii. *if S is extremally disconnected, then $C^\infty(S)$ is a Dedekind complete Riesz space.*

Finally, unbounded increasing sequences in $C^\infty(S)$ have a property that we will be able to use later.

#Lemma 0.12 (Fremlin)
Let S be a basically disconnected compact Hausdorff space.
Suppose that $0 \le u_n \uparrow$ in $C^\infty(S)$. Then either $\sup_n u_n$ exists in $C^\infty(S)$ or there exists a $w^* \in C^\infty(S)$, $0 < w^*$, such that for all $r \in (0, \infty)$ and all $0 \le w \le w^*$: $\sup_n r u_n \wedge w = w$.
If S is extremally disconnected the conclusion holds for upwards directed nets as well.

Lemma 0.12 can be smoothly proved using the concept of semi-continuity of functions $S \to [-\infty, \infty]$. For completeness, we will provide such a proof, and since it is almost no extra effort, we will prove 0.11 and 0.10 along the way.

A function $f : S \to [-\infty, \infty]$ is called upper (lower) semi-continuous if the sets $[f < r]$, $r \in [-\infty, \infty]$, (the sets $[f > r]$, $r \in [-\infty, \infty]$, respectively) are open. A function is continuous if and only if it is both upper and lower semi-continuous. A supremum of a collection of lower semi-continuous functions is itself lower semi-continuous. A sum of upper (lower) semi-continuous functions is again upper (lower) semi-continuous.
Given $f : S \to [-\infty, \infty]$, we define $f^{\uparrow}, f^{\downarrow} : S \to [-\infty, \infty]$ by

$$\left.\begin{array}{l} f^{\uparrow}(a) := \inf_{V \in \tau_a} \sup_V f \,, \\ f^{\downarrow}(a) := \sup_{V \in \tau_a} \inf_V f \,, \end{array}\right\} \quad (a \in S) \,,$$

where τ_a is a neighborhood base of $a \in S$. One verifies elementarily that these definitions do not depend on the neighborhood basis chosen, that $f^{\uparrow}$ is the smallest upper semi-continuous function that lies above f, and $f^{\downarrow}$ is the greatest lower semi-continuous function that lies below f (cf. [dJvR, 12H p. 87]).
Also, if f is lower semi-continuous, then $f^{\uparrow} = f$ on a dense set, because for all $n \in \mathbb{N}$, $[f^{\uparrow} > f + n^{-1}]$ has empty interior (use Urysohn's lemma and the minimality of $f^{\uparrow}$).
Finally, observe that

$$[f^{\uparrow} > r] = \cup_{k \in \mathbb{N}} \overline{[f > r + k^{-1}]} \quad (r \in [-\infty, \infty] \,,\ f : S \to [-\infty, \infty]) \,,$$

so that $f^{\uparrow}$ is continuous whenever all the sets $\overline{[f > t]}$, $t \in [-\infty, \infty]$, are open.

Now, both 0.10 and 0.11 are an application of the latter observation:

For 0.10 extend f to $\overline{O}$ by setting $f = -\infty$ on $\overline{O} \setminus O$. Then $\overline{f} := f^{\uparrow}$ is continuous, because $[f > r]$ has clopen closure for all $r \in [-\infty, \infty]$.

For proving 0.11, let $\mathcal{F}$ be a collection of continuous functions $S \to [-\infty, \infty]$, and let f be the pointwise supremum of $\mathcal{F}$. In case $\mathcal{F} = \{f_n\}_n$ is countable we have that $[f > t] = \cup_{n \in \mathbb{N}} [f_n > t] = \cup_{n,m \in \mathbb{N}} [f_n \geq t + m^{-1}]$ is an open F_σ, so has an open closure if S is basically disconnected.
In case $\mathcal{F}$ is not countable, $[f > t] = \cup_{g \in \mathcal{F}} [g > t]$ is open and has an open closure (for all $r \in [-\infty, \infty]$) provided S is extremally disconnected.
In both cases, $f^{\uparrow}$ is the supremum of $\mathcal{F}$ in the lattice of all continuous functions $S \to [-\infty, \infty]$. Similarly, if h is the pointwise infimum of $\mathcal{F}$, then $h^{\downarrow}$ is the infimum of $\mathcal{F}$. Were now $\mathcal{F}$ bounded from above (below) by an element of $C^\infty(S)$, then so were $f^{\uparrow}$ ($h^{\downarrow}$), which establishes 0.11.

Proof of 0.12
Let $u := u^{\downarrow}_\infty$, where $u_\infty(s) := \sup_n u_n(s)$ ($s \in S$), be the supremum of $\{u_n\}_n$ in the pointwise ordered collection of continuous functions $S \to [-\infty, \infty]$.
Case $[u = \infty]^\circ = \emptyset$. Then $C^\infty(S) \ni u$ is the supremum of $\{u_n\}_n$ in $C^\infty(S)$.

Case $U := [u = \infty]^\circ \neq \emptyset$. Observe that U is clopen, being the complement of the closure of the open F_σ-set $[u < \infty] = \cup_{n \in \mathbb{N}} [u \leq n]$.
Let $0 < w \in C^\infty(S)$ with $[w > 0] \subset U$ and let $r \in (0, \infty)$. We will show that $\sup_n r u_n \wedge w = w$. In particular, we could take $\mathbb{1}_U$ for w^*.

Since w is already an upper bound of $\{ru_n \wedge w\}_n$, we only need to show it is the smallest. To this end, let $z \in C^\infty(S)$ such that $z \geq ru_n \wedge w$ for all $n \in \mathbb{N}$. Then $z(s) \geq ru_\infty(s) \wedge w(s)$ for all $s \in S$. Further, since $u = u_\infty$ on a dense set, $z \geq ru \wedge w = w$ on a dense set of S, so $z \geq w$ in $C^\infty(S)$. □

We now introduce the theory of Riesz spaces that will play a rôle in the sequel. We advise the reader to think of elements of a Riesz space as (extended) real-valued functions (on some fixed topological space, see 0.29 and 0.30 further on), because this gives in general a good intuitive idea of the concepts and the kind of results one can obtain.

For the rest of this section (i.e. up to 0.4 on page 27), E and F will denote Riesz spaces.

0.3.2 Elementary inequalities

We start with some inequalities that will be used without reference.

Lemma 0.13 ([dJvR, 1.4, p. 8], [LuZa, p. 55-66])

In the following f, g, h denote elements of E, G denotes a subset of E, and r denotes a real number. Further, we set $rG := \{rg : g \in G\}$, and $f + G = G + f$ will stand for the set $\{f + g : g \in G\}$.

1. *if $r \in [0, \infty)$: $r(f \vee g) = rf \vee rg$ and $r(f \wedge g) = rf \wedge rg$; in fact, $r \sup G = \sup(rG)$ and $r \inf G = \inf(rG)$ if the supremum (infimum) of G exists;*
2. *if $r \in (-\infty, 0]$: $r(f \wedge g) = (rf) \vee (rg)$ and $r(f \vee g) = (rf) \wedge (rg)$; in fact, $r \sup G = \inf(rG)$ and $r \inf G = \sup(rG)$ if the supremum (infimum) of G exists;*
3. *$-f^- \leq f \leq f^+$;*
4. *$|f + g| \leq |f| + |g|$, $(f + g)^+ \leq f^+ + g^+$,*
   in particular, $(u - v)^+ \leq u^+ = u$ if $0 \leq u, v \in E$;
5. *$f = f^+ - f^-$, and $f^+ \wedge f^- = 0$; in fact, if $f = u - v$ where $0 \leq u, v \in E$ with $u \wedge v = 0$, then $u = f^+$ and $v = f^-$;*
6. *$|f \vee h - g \vee h| + |f \wedge h - g \wedge h| = |f - g|$ (Birkhoff's identity);*
7. *$|f \vee h - g \vee h| \leq |f - g|$ and $|f \wedge h - g \wedge h| \leq |f - g|$ (Birkhoff's inequalities);*
8. *$f + g = f \vee g + f \wedge g$;*
9. *$(f + g) + |f - g| = 2(f \vee g)$ and $f + g - |f - g| = 2(f \wedge g)$;*
10. *$\sup(f + G) = f + \sup G$ and $\inf(f + G) = f + \sup G$ if either side exists (infinite distributivity);*
11. *$f \wedge \sup G = \sup\{f \wedge g : g \in G\}$ if the supremum of G exists; the same holds with the supremum and infimum interchanged;*
12. *if $|h| \leq |f| + |g|$, then there exist $h_f, h_g \in E$ such that $h = h_f + h_g$ and $|h_f| \leq |f|$ while $|h_g| \leq |g|$. (Riesz decomposition property).*

In fact: any (in)equality involving finitely many elements $f, g, h, \ldots$ and Riesz space operations $(+, \cdot, \wedge, \vee, {}^+, {}^-)$ that holds if $f, g, h, \ldots$ are elements of $\mathbb{R}$, holds in any Riesz space (cf. 0.32).

Indeed, such (in)equalities can be proven elementarily (starting from the definitions) or by representing the elements involved as real-valued functions (see 0.29 and 0.30 later on).

An important thing to note at this stage is that every element in a Riesz space is the difference of two positive ones, which often helps to reduce a statement concerning a Riesz space to statement involving its positive cone only. E.g.:

Lemma 0.14
E is (σ-)Dedekind complete if and only if every non-empty majorized, (countable) upwards directed subset in E^+ has a supremum.
In particular, in view of lemma 0.4, L^p is Dedekind complete.

Indeed, if G is a non-empty subset of E (say $g_0 \in G$), then G has the same upper bounds as $G^\vee := \{g_1 \vee \ldots \vee g_n : n \in \mathbb{N}, g_i \in G\}$; further, h is an upper bound of $G^\vee$ if and only if $h - g_0 = (h - g_0)^+$ is an upper bound of $\{(g - g_0)^+ : g \in G^\vee\}$.

0.3.3 Riesz subspaces, Ideals and Bands

Given a Riesz space, we consider three types of subspaces that are Riesz spaces in their own right when endowed with the restrictions of the Riesz space operations.

Definition 0.15 (Riesz subspaces)
A linear subspace E_0 of E is called a Riesz subspace of E if $f, g \in E_0$ implies $f \vee g \in E_0$.

Let E_0 be a Riesz subspace of E. In view of the elementary inequalities discussed above, $f \wedge g$, f^+, $|f|$ etc. are in E_0 whenever f and g are.

Example 0.16
- $C[0,1]$ is a Riesz subspace of $L^p[0,1]$.
- If we endow the polynomials on $[0,1]$ of degree less than or equal to one with the pointwise ordering, then they form a Riesz space that is not a Riesz *sub*space of $C[0,1]$.

The next two types of Riesz subspaces are closed in a stronger sense with respect to the ordering.

Definition 0.17 (Ideals)
A subset S of E is called solid if $f \in E$, $g \in S$, and $|f| \le |g|$ implies that $f \in S$.
A solid linear subspace of E is called an (order) ideal of E.
The intersection of a collection of ideals in E is again an ideal, and so is E itself. Therefore, given $A \subset E$, there exist a smallest ideal that contains A. This ideal, denoted by $E_{[A]}$, is called the ideal generated by A.
In particular, given $e \in E$, the ideal generated by $\{e\}$, is called the principal ideal generated by e. It is denoted by $E_{[e]}$.

Examples 0.18
- An ideal is indeed a Riesz subspace.
- $E_{[A]} = \{f \in E : \exists\, a_1, \ldots, a_n \in A,\ K \in \mathbb{N}$ such that $|f| \le K \cdot \sum_1^n |a_i|\}$.
- In particular, $E_{[e]} = \{f \in E : \exists n \in \mathbb{N}$ such that $|f| \le n|e|\}$.
- $C(S)$ is an ideal of $C^\infty(S)$.

- For fixed $x_0 \in [0,1]$, $\{f \in C(S) : f(x_0) = 0\}$ is an ideal of $C[0,1]$.
- $\{f \in C[0,1] : f(0) = f(1)\}$ is not an ideal of $C[0,1]$, but it is a $\| \|_\infty$-closed Riesz subspace of $C[0,1]$.

Definition 0.19
An element $e \in E$ is called an (order) unit or strong unit of E, if for every $f \in E$ there exists an $n \in \mathbb{N}$ such that $|f| \le n|e|$.

$\mathbb{1}$ is a unit of $C(S)$. Tautologically, every principal ideal of E has a unit.

Definition 0.20 (Bands)
An ideal of B of E is called a band if $\sup D \in B$ *for every subset $D \subset B$ for which* $\sup D$ *exists in E.*
The intersection of a collection of bands is again a band and so is E itself. Therefore, given a subset A of E, it makes sense to speak about the smallest band containing A. This band is called the band generated by A, notation $E_{<A>}$.
Given $e \in E$, the band generated by $\{e\}$ is called the principal band generated by e, and it is denoted by $E_{<e>}$.

Example 0.21
- $u \in E^+_{<e>}$ if and only if $u \wedge n|e| \uparrow u$. ($E^+_{<e>}$ refers here to the positive cone of the Riesz space $E_{<e>}$).
- The ideal $\{f \in C[0,1] : f(0) = 0\}$ of $C[0,1]$ is no band, because $\mathbb{1} = \sup_n \mathbb{1} \wedge n\varkappa$, where $\varkappa(t) = t$ $(t \in [0,1])$.

Example 0.22 (Bands in $L^p(\mu)$)
Let $(S, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $p \in (0, \infty)$. For an $A \in \mathcal{A}$, $B_A := \{f : f = 0\ \mu\text{-almost everywhere on } S \setminus A\}$ is a band. Conversely, $B \subset L^p(\mu)$ is a band if and only if there is an $A \in \mathcal{A}$ with $B = B_A$ (proof is like that in [dJvR, 4I, p. 30]).

Example 0.23 (Bands in $C(S)$)
Let S be a compact Hausdorff space. For every non-empty open $U \subset S$, the collection $B_U := \{f : [f \neq 0] \subset U\}$ is a band. Conversely, if B is a band of $C(S)$, then there exists a non-empty open subset U such that $B = B_U$ (e.g. $U = \cup_{v \in B^+}[v > 0]$), but this U is not unique, since $B_U = B_V$ as soon as $\overline{U} = \overline{V}$. We can force uniqueness by requiring U to be regular open i.e. $\overline{U}^\circ = U$. Thus, there is a one-to-one correspondence between bands of $C(S)$ and regular open subsets of S ([dJvR, Thm. 12.9, p. 83]).

Definition 0.24
An element e of E^+ is called a weak unit of E if $f \in E$, $|f| \wedge e = 0$ implies $f = 0$.

$\mathbb{1}$ is a weak unit of $L^p[0,1]$. An element in $E = C^\infty(S)$ or $E = L^p(\mu)$ is a weak unit if and only if the band it generates coincides with E.

We end this section on Riesz subspaces with a caveat: if E_0 is a Riesz subspace of E and $F \subset E_0$, then E_0-$\sup F = E$-$\sup F$ if F is finite, but this need not be so if F is infinite: e.g. the supremum of $\{n\varkappa \wedge \mathbb{1} : n \in \mathbb{N}\}$ in $C[0,1]$ is $\mathbb{1}$, but in $\mathbb{R}^{[0,1]}$ it is $\mathbb{1} - \mathbb{1}_{\{0\}}$.

0.3.4 Spectral representations of a Riesz space

We return to our remark that "elements of a Riesz space" can be thought of as (extended) real-valued functions on a fixed topological space.
First, the way we identify a Riesz space with a space of functions is via a so-called Riesz isomorphism (following the terminology of [AlBu, p. 8] an isomorphism does not have to be surjective).

Definition 0.25
A linear map ϕ from E into another Riesz space is called a Riesz homomorphism if $\phi(f \wedge g) = \phi(f) \wedge \phi(g)$. If it is in addition injective (but not necessarily surjective), it is called a Riesz isomorphism.

In view of the elementary inequalities discussed before, a Riesz homomorphism preserves all Riesz space operations, and its image is a Riesz (sub)space.

Definition 0.26
We say that E has a spectral representation if there exist a topological space S and a Riesz isomorphism from E onto a Riesz space of extended real-valued functions on S (p. 8).

All Riesz spaces that allow a spectral representation have the following property in common:

Definition 0.27
E is said to be Archimedean if for all $u, v \in E^+$: $0 \le u \le n^{-1}v$ $(n \in \mathbb{N})$ implies that $u = 0$.

Examples 0.28
The Riesz space consisting of $\mathbb{R}^2$ equipped with the lexicographical ordering, i.e. $(x, y) \le (x', y')$ if $x < x'$ or else $x = x'$ and $y \le y'$, is not Archimedean, because $(0,0) \le (0,1) \le n^{-1}(1,0)$ for all $n \in \mathbb{N}$.

We will deal with Archimedean Riesz spaces only, and we will frequently rely on one of the two following representation theorems (see [dJvR, 13.11, p. 96] and [dJvR, Thm. 15.5, p. 123] respectively).

Theorem 0.29 (Yoshida's representation theorem)
Let E be Archimedean and suppose that E has a unit e. Then

$$||f||_e := \inf\{r \in (0,\infty) : |f| \le re\}$$

defines a norm on E. Further, there exist a compact Hausdorff space Ω and a Riesz isomorphism $\hat{\ }$ from E into $C(\Omega)$ such that

$$\hat{e} = \mathbb{1}\ , \ \|\hat{f}\|_\infty = ||f||_e \quad (f \in E)\ , \ \text{ and } \hat{E} \text{ is } \|\ \|_\infty\text{-dense in } C(\Omega).$$

In particular, if E is complete with respect to $\|\ \|_e$, then $\hat{E} = C(\Omega)$.
The space $C(\Omega)$ is called a Yoshida representation space of E.

Theorem 0.30 (Maeda-Ogasawara-Vulikh's representation theorem)
Let E be Archimedean.
Then there exist an extremally disconnected compact Hausdorff space S and a Riesz isomorphism Φ from E into $C^\infty(S)$.
Furthermore, identifying E with the Riesz subspace $\Phi(E)$ of $C^\infty(S)$, we have for all $u \in C^\infty(S)^+$: $u = \sup\{v \in E^+ : 0 \le v \le u\}$.

The norm denseness in 0.29 has an interesting consequence.

Lemma 0.31
Let S be a compact Hausdorff space and let E be a $\| \|_\infty$-dense Riesz subspace of $C(S)$ containing $\mathbb{1}_S$.

i. *If $S_0, S_1 \subset \Omega$ are disjoint and closed, then there exists an $f \in E$ that is 1 on S_1 and 0 on S_0 (a so-called Urysohn-function for S_0 and S_1).*
ii. *In particular, E contains all indicators of clopen subsets of S.*

◁Indeed, Urysohn's lemma implies that there is a $g \in C(S)$ such that g is -1 on S_0 and 2 on S_1. Approximate g by $f_1 \in E$ such that $\| g - f_1 \|_\infty < 1$, and set $f := f_1^+ \wedge \mathbb{1} \in E^+$.▷

Another consequence of Yoshida's theorem is

Lemma 0.32 (Yudin)
Let E be a Riesz space having vector space dimension $n < \infty$. Then E is Riesz isomorphic to $\mathbb{R}^n$ (with the coordinatewise ordering).

◁Indeed, let $(e_1, \ldots, e_n)$ be a vector space basis of E. Then $e := |e_1| + \ldots + |e_n|$ is a strong unit of E, and being a finite dimensional vector space, E is complete with respect to the norm $\| \|_e$, so $\hat{E} = C(S)$. Since $\dim(C(S)) = n$, we have that S has n elements.▷

0.3.5 Disjointness

Many properties of collections of functions have a "translation" in terms of Riesz space structure. We begin with a concept that expresses that functions have disjoint support.

Definition 0.33
Let D be a subset of E.
We say that f and g are disjoint, notation $f \perp g$, if $|f| \wedge |g| = 0$. We say that f is disjoint from D, notation $f \perp D$, if f is disjoint from all elements of D. Similarly, $B \subset E$ is disjoint from D, notation $B \perp D$, whenever every element of B is disjoint with every element of D.
D itself is called disjoint if the elements of D are pairwise disjoint.
The disjoint complement of D, denoted by $D^\perp$ or D^d, is the set $\{g \in E : g \perp D\}$.
If $x_1, \ldots, x_n$ are disjoint elements with sum x, then we will often write

$$\textstyle\sum_1^n x_i = x \quad \text{disjoint.}$$

As intuitively expected, we have properties like:

1. If $f, g \in C^\infty(S)$, then f and g are disjoint if and only if $[f \neq 0] \cap [g \neq 0] = \emptyset$.

   If $f, g \in L^p(\mu)$, then f and g are disjoint if and only if $[f \neq 0] \cap [g \neq 0]$ is a negligible set.
2. $f \perp g$, $|h| \leq |f|$ implies $h \perp g$ (so $D^\perp$ is an ideal);
3. Let $B \subset E$. If $\sup B$ exists, then $f \perp B \Rightarrow f \perp \sup B$ (by infinite distributivity).

   Consequently, if $\sup B$ exists, then $B \perp D \Rightarrow \sup B \perp D$ (so $D^\perp$ is a band).
4. $D \cap D^\perp = \{0\}$.
5. If $u_1, \ldots, u_n$ are disjoint positive elements, then $u_1 \vee \ldots \vee u_n = u_1 + \ldots + u_n$.

If $n = \dim(E) < \infty$, then there exists a disjoint sequence of n non-zero vectors in E (use 0.32). Similarly, if E is infinite dimensional, then there exists an infinite disjoint sequence of non-zero vectors in E, as we will see in 0.42.

We mention a lemma that will be of use later.

#Lemma 0.34 (Half-disjointness)

Let E be Archimedean. Let $u, e \in E^+$, and let $0 =: \alpha_0 < \alpha_1 < \ldots \uparrow \infty$ in $[0, \infty)$. Then there exists a sequence $(u_n)_{n=1}^{\infty}$ in the Riesz subspace generated by u and e such that

i. $0 \le u_n \le \alpha_{n+1} \cdot e$ *and* $0 \le u_n \le u$ *for all* n;
ii. $\{u_n\}$ *has the same upper bounds as* $\{u \wedge ke : k \in \mathbb{N}\}$ *in* E;
iii. $u_n \wedge u_m = 0$ *if* $|n - m| \ge 2$ *(we therefore say that* $(u_n)_n$ *is half-disjoint);*
iv. $\alpha_{n+1} \cdot [e \wedge (ku_n)] \le u + (\alpha_{n+1} - \alpha_{n-1}) \cdot e$ *for all* $n, k \in \mathbb{N}$.

Proof

Set for $n = 1, 2, \ldots$

$$u_n := \frac{(u - \alpha_{n-1} \cdot e)^+}{(\alpha_n - \alpha_{n-1})/(\alpha_n + \alpha_{n-1})} \wedge u \wedge \frac{(u - \alpha_{n+1} \cdot e)^-}{(\alpha_{n+1} - \alpha_n)/(\alpha_{n+1} + \alpha_n)}$$

and check the claims via a Yoshida representation that contains e and u. □

#0.3.6 Band projections and components

Definition 0.35

Let E be a Riesz space. A projection band is a band B such that $B + B^\perp = E$.
Let B be a projection band. The linear map P that maps $f \in E$ to $Pf \in B$ such that $f - Pf \perp B$, is called the band projection onto B, notation P_B^E or shortly P_B. If B is generated by one element of E, then B is called a principal projection band, and the corresponding projection is denoted by P_e^E or P_e for short.
We say that E has the (principal) projection property if every (principal) band is a projection band.

A more explicit description of band projections can be given [LuZa, 24.5,24.7 (p.133-135]):

Lemma 0.36

Let $e \in E$, and let B be a band of E.

i. *Take $u \in E^+$.*

   If $u_e := \sup\{u \wedge n|e| : n \in \mathbb{N}\}$ exists, then $u_e \in E_{<e>}$ and $u - u_e \perp E_{<e>}$.

   If $u_B := \sup\{v \in B : 0 \le v \le u\}$ exists, then $u_B \in B$ and $u - u_B \perp B$.
ii. *$E_{<e>}$ is a principal projection band if and only if $\sup_n\{u \wedge n|e|\}$ exists for all $u \in E^+$, and in that case $P_e(u) = \sup_n u \wedge n|e|$.*

   The band projection onto B exists if $\sup\{v \in B : 0 \le v \le u\}$ exists for all $u \in E^+$, and in that case $P_B(u) = \sup\{v \in B : 0 \le v \le u\}$.
iii. *Therefore: if E is (σ-)Dedekind complete, then E has the (principal) projection property.*

Examples 0.37

1. Let S be a compact Hausdorff space. Then the projection bands in $C(S)$ correspond to clopen subsets (via the identification 0.23). As a result, in general not every band is a projection band.

   The band projection onto $B = \{h \in C(S) : [h \neq 0] \subset U\}$, where U is clopen, is given by $P_B(f) = f\mathbb{1}_U$ $(f \in C(S))$.

   In particular, the principal band generated by $b \in C(S)^+$ is a (principal) projection band if and only if $\overline{[b > 0]}$ is clopen. As a result, $C(S)$ has the principal projection property if and only if the sets $\overline{[u > 0]}$, $u \in C(S)^+$, are all clopen (i.e. S is basically disconnected).
2. In $L^p(\mu)$, every band is a projection band, because L^p is Dedekind complete.
3. Every finite dimensional ideal (in an Archimedean Riesz space) is a projection band.

   ◁ Indeed, if $e \in E^+$, and $E_e = \mathbb{R} \cdot e$ is a one-dimensional ideal, then for every $u \in E^+$, $R_u := \{r \in (0, \infty) : re \leq u\}$ is bounded in $\mathbb{R}^+$ (by the Archimedean property), so that $P_e(u) = \sup\{v \in E_e^+ : 0 \leq v \leq u\} = (\sup R_u) \cdot e$ exists.
   Further, a n-dimensional ideal I of E is Riesz isomorphic to $\mathbb{R}^n$, so that I is a sum of disjoint one-dimensional ideals $E_{e_1}, \ldots, E_{e_n}$. Correspondingly, I is a projection band with $P_I = P_{e_1} + \ldots + P_{e_n}$.▷

From the description of band projections we elementarily obtain:

Lemma 0.38

Let P be a band projection onto the band B of E. Then:

i. $0 \leq P(u) \leq u$ *if* $u \in E^+$ *(P is an order bounded map);*

ii. $P(u) = 0$ *if and only if* $u \perp B$*; as a special application:* $P_{(u-v)^+}((u - v)^-) = 0$*;*

iii. P *is a Riesz homomorphism (because* $u \wedge v = 0 \Rightarrow P(u \wedge v) = 0$*);*

iv. $P(\sup A) = \sup_{a \in A} P(a)$ *(use infinite distributivity 0.13).*

For principal band projections we can add the following property:

$$P_v(u) \leq P_w(u) \text{ if } 0 \leq v \leq w \text{ , and } 0 \leq u \text{ in } E .$$

We now turn to the concept of components.

Definition 0.39

An element $g \in E$ is called a component of $f \in E$ if $f - g$ and g are disjoint.
The collection of all components of f in E is denoted by $C_E(f)$ or $C(f)$ for short.
If $g_1, g_2 \in C_E(f)$, then we refer to the fact that $g_1 + g_2 = f$ by saying that g_1 is the complement of g_2 (and vice versa).

Examples 0.40

- If $f \in L^p(S, \mathcal{A}, \mu)$, then $C(f) = \{f\mathbb{1}_A : A \in \mathcal{A}\}$.
- If $f \in C(S)$ is a weak unit (i.e. f has no zeroes), then

$$C(f) = \{f\mathbb{1}_U : U \text{ clopen}\} .$$

In particular, if E is an Archimedean Riesz space, and $f \in E$, then by taking a Yoshida representation of $E_{[f]}$ we can identify $C_E(f)$ with $\{\mathbb{1}_U : U \text{ clopen in } E\}$.

- The non-trivial components of $t \mapsto |t|$ in $C[-1,1]$ are $t \mapsto t^+$ and $t \mapsto t^-$.

We mention some properties of components that will be of use later.

Lemma 0.41

Let $e \in E^+$.

i. *$C_E(e)$ is a lattice.*
ii. *The supremum or infimum of any collection of components of e is a component itself (use the infinite distributivity). In fact, if C is a collection of components of e, then $\sup C = e - \inf\{e - u : u \in C\}$.*
iii. *If $w \geq v \geq 0$ in $C(e)$, then $w - v$ is a component of e, disjoint with v (e.g. use a Yoshida representation). In particular: if $v \in C(e)$, then so is the so-called complement of v: $e - v$.*

We can use projections in the following:

Lemma 0.42

Let E be an infinite dimensional Archimedean Riesz space. Then there exists an infinite disjoint sequence of non-zero elements in E.

$\lhd$ For $e \in E^+$ we denote the Yoshida representation space of $E_{[e]}$ by Ω_e We distinguish two cases·

Case there exists an $e \in E^+$ such that Ω_e is infinite In that case, Urysohn's lemma helps us to find a infinite disjoint sequence of Urysohn-functions $(e_n)_n$ in $C(\Omega_e)$, and by 0 31 we can arrange that $(e_n)_n$ in E_e^+

Case Ω_e is finite for all $e \in E^+$. Then every principal ideal is finite dimensional, hence (0.37) a projection band (with infinite dimensional disjoint complement).

Inductively select $(e_n)_n$ such that $0 < e_{n+1} \in (E_{[e_1 + \quad + e_n]})^d$ $\rhd$

#0.3.7 Order related convergence and continuity concepts

We discuss two types of convergence and continuity that are compatible with the Riesz space ordering.

Definition 0.43 (Relative uniform convergence)

Let $(f_n)_n$ be a sequence in E, let $e \in E^+$, and let $\varepsilon_k \in (0, \infty)$ such that $\varepsilon_k \downarrow 0$.
If $|f_n - f_m| \leq \varepsilon_k e$ $(n, m \geq k)$, then we say that $(f_n)_n$ is e-uniformly Cauchy.
If there exists an $f \in E$ such that $|f - f_n| \leq \varepsilon_k e$ $(n \geq k)$, then we say that $(f_n)_n$ is e-uniformly convergent to f, notation $f_n \xrightarrow{e} f$. In this case, we refer to e as a regulator of convergence.

We say that $(f_n)_n$ is relatively uniformly convergent, (r.u.-convergent) or relatively uniformly Cauchy, r.u.-Cauchy if it is e-convergent or e-Cauchy for some $e \in E^+$.

A subset $A \subset E$ is called relatively uniformly closed (r.u.-closed) if $e \in E^+$, $f \in E$, $(f_n)_n$ in A and $f_n \xrightarrow{e} f$ implies that $f \in A$.
The r.u.-closed subsets of E form the collection of closed sets of a topology, which is called the relatively uniform topology.

Example 0.44

In $C(S)$, $\mathbb{1}$-uniform convergence is the usual uniform convergence. In fact, relatively uniform convergence coincides with the usual uniform convergence, and therefore the uniform topology is the topology generated by $\| \|_\infty$.

We introduce a weaker form of convergence that we will only formulate for the context of (σ-)Dedekind complete Riesz spaces.

Definition 0.45 (Order convergence)

Let E, F be Dedekind complete Riesz spaces. Let $(f_\alpha)_\alpha$ be a net in E, and $f \in E$.

We say that f_α is order convergent to f, notation $f_\alpha \overset{(o)}{\to} f$, if there exists a net $(p_\alpha)_\alpha$ in E such that $p_\alpha \downarrow 0$ and $|f - f_\alpha| \le p_\alpha$ for all α.

A map $\Phi : E \to F$ is called order continuous if it preserves order convergence.

A subset $A \subset E$ is called order closed if $(f_\alpha)_\alpha$ in A, $f \in E$, and $f_\alpha \overset{(o)}{\to} f$ implies that $f \in A$.

By substituting "sequences" for "nets" and "σ-Dedekind" for "Dedekind" in the above, we define σ-order continuity and σ-order closedness.

Examples 0.46

- If $\Phi : E \to F$ is positive (i.e. $\Phi(u) \ge 0$ if $u \ge 0$), then Φ is order continuous if and only if $u_\alpha \downarrow 0$ implies $\Phi(u_\alpha) \downarrow 0$ (because $0 \le |\Phi(f - f_\alpha)| \le \Phi|f - f_\alpha| \le \Phi(p_\alpha)$).
- If $(u_\alpha)_\alpha$ is upwards directed to u, then $u_\alpha \overset{(o)}{\to} u$.
- Bands are order closed ideals.
- Band projections are order continuous Riesz homomorphisms: if P is a band projection then $|P(f_\alpha) - P(f)| \le P(|f_\alpha - f|) \le |f_\alpha - f|$.
- The function $\|\ \|_p : L^p(\mu) \to \mathbb{R}$ is σ-order continuous (use e.g. $\|\ \|_p$-inequalities and Lebesgue's dominated convergence theorem). Using that L^p has the countable sup property (see 0.58 further on), one can see that $\|\ \|_p$ is in fact order continuous.

0.3.8 Special properties of Riesz (sub)spaces

We discuss some additional Riesz space features, most of which are illustrated by L^p-spaces.

Let E_0 be a Riesz subspace of E.

A subset of E_0 may have a supremum in E_0, but not in E, or vice versa. Even if both suprema exist, they may not be the same (although the E-supremum is at most equal to the E_0-supremum).

Definition 0.47

If every subset $A \subset E$ that has a supremum in E_0 also has a supremum in E such that E_0-$\sup A = E$-$\sup A$, then we say that E_0 is a regular Riesz subspace of E.

A sufficient condition for E_0 to be regular in E is the following:

Definition 0.48

E_0 is said to be order dense in E if for all $0 < u \in E$, there exists a $u_0 \in E_0$ such that $0 < u_0 \le u$.

Equivalently: $u = \sup\{v_0 \in E_0 : 0 \le v_0 \le u\}$ for all $u \in E^+$.

In particular, if E_0 is order dense in E, then for every element in E^+ there exists a system in E_0^+ that is upwards directed to it. A bit stronger is the following:

#Definition 0.49

E_0 is said to be super-order dense in E if for all $0 < u \in E$, there exists a sequence $(u_n)_n$ in E_0 with $0 \le u_n \uparrow u$.

Examples 0.50

- The Riesz subspace $\{r\mathbb{1} : r \in \mathbb{R}\} \simeq \mathbb{R}$ of $C[0,1]$ is regular but not order dense (in $C[0,1]$).
- The Riesz space $\{f \in L^p : f$ has a representative that is a step function$\}$ is super-order dense in L^p.
- The Riesz subspace $C[0,1]$ of $\mathbb{R}^{[0,1]}$ is not regular (see the caveat after 0.24).
- A $\|\ \|_\infty$-dense Riesz subspace of $C(S)$ that contains $\mathbb{1}$ is order dense (0.31).

Definition 0.51

E_0 is called a majorizing Riesz subspace of E (a full Riesz subspace in the terminology of [AlBu, Def. 1.7, p. 6]) if for every $v \in E^+$ there exists a $v_0 \in E_0$ that majorizes u i.e. $v \le v_0$.

Example 0.52

The Riesz subspace $\{r\mathbb{1} : r \in \mathbb{R}\}$ is majorizing in $C[0,1]$, but not in $L^p[0,1]$.

The combination of having E_0 order dense and majorizing in E is convenient because then every element in E^+ is the supremum of all elements in E_0^+ below it, and the infimum of all elements in E_0^+ above it. In particular, for every $x \in E$ there is a net $(x_\alpha)_\alpha$ in E_0 that order converges to x.
The properties that remain to be mentioned are "local" in the sense that they address the principal ideals of E rather than E as a whole.

Definition 0.53

We say that E is uniformly complete if every relatively uniformly Cauchy sequence in E is relatively uniformly convergent to a member of E.

In fact, uniform completeness of E comes down to the fact that for all $e \in E^+$, the principal ideal $E_{[e]}$ is complete with respect to $\|\ \|_e$. This means (see 0.29):

Corollary 0.54

If E is uniformly complete, then every principal ideal is Riesz isomorphic to a $C(S)$ for some compact Hausdorff space S.

Examples 0.55

- The spaces $C(S)$, with S compact Hausdorff, are uniformly complete;
- the Riesz space $\mathbb{R} + c_{00} = \{x \in c : x$ takes only finitely many values$\}$ is not: the sequence $x_n := \sum_{i=1}^n i^{-1}e_i$, $n \in \mathbb{N}$, is $\mathbb{1}$-Cauchy, but not relatively uniformly convergent in $\mathbb{R} + c_{00}$.

A simple criterion to check uniform completeness of E is:

Lemma 0.56 (Criterion for uniform completeness)

E is uniformly complete if and only if for all $u \in E^+$, and $(u_n)_n$ in E^+

$$0 \le u_n \le 2^{-n}u \quad \Longrightarrow \quad E\text{-}\sup_n u_n \text{ exists}.$$

In particular, σ-Dedekind complete Riesz spaces are uniformly complete.

The next property we discuss is a countability condition introduced by Fremlin.

#Definition 0.57

We say that E has the countable sup property if every set that has a supremum in E contains a countable subset having the same supremum (the same then holds for infima).
If E is in addition Dedekind complete, then E is called super-Dedekind complete.

L^p-spaces have the countable sup property: this follows from 0.4 or from the following lemma ([Fr, 18D, p. 31]) and the p-additivity of $\| \|_p$.

#Lemma 0.58 (Criterion countable sup property)
E has the countable sup property if and only if every majorized disjoint collection in E^+ is countable.

#Examples 0.59

- L^p-spaces are super-Dedekind complete.
- If S is a compact Hausdorff space, then $C(S)$ has the countable sup property, whenever every collection of disjoint open sets is countable. A sufficient condition for that is separability of S, but this condition is not necessary: if $\mathbb{A}$ is a set having a cardinality greater than the continuum, then the topological group $S = \{0,1\}^{\mathbb{A}}$ supports a Haar measure (so that every collection of disjoint open sets is countable), but S is not separable.

  ◁Indeed: *were* $\{s_n\}_{n\in\mathbb{N}} \subset S$ dense, then with $\mathcal{I}_\alpha := \{n \in \mathbb{N} : s_n(\alpha) = 1\}$ a cardinality argument would yield that the map $\alpha \mapsto \mathcal{I}_\alpha$ could not be injective, whence there would be $\alpha \neq \beta$ in $\mathbb{A}$ such that $\mathcal{I}_\alpha = \mathcal{I}_\beta$. Then $s_n(\alpha) = s_n(\beta)$ for all $n \in \mathbb{N}$, so $s(\alpha) = s(\beta)$ for all $s \in S$. *Contradiction.*▷

- The space $\mathbb{R}^S$ with S uncountable fails to have the countable sup property.

Definition 0.60
E is called conditionally σ-laterally complete if every majorized disjoint countable subset of E^+ has a supremum in E.

Examples 0.61

- σ-Dedekind complete Riesz spaces (such as L^p) are conditionally σ-laterally complete.
- $C[0,1]$ is not conditionally σ-laterally complete: if $0 \leq f_n \leq \mathbb{1}$ are such that $f_n((2n)^{-1}) = 1$ and $f_n = 0$ outside $((2n+1)^{-1}, (2n-1)^{-1})$, then $(f_n)_n$ has no supremum in $C[0,1]$: consider the situation at 0.

We conclude with the property that is the pivot on which chapter 1 (1.2) turns: the weak-Freudenthal property. As far as we know this notion was first introduced by Lavrič ([La]). Basically, we can describe the weak-Freudenthal property as follows: by Yoshida's theorem (0.29) there exists for every principal ideal I of E a compact Hausdorff space S_I such that I is Riesz isomorphic to a $\| \|_\infty$-dense Riesz subspace of $C(S_I)$; if now for every principal ideal I of E, S_I is zero-dimensional (i.e. its topology has a basis of clopen subsets), then we say that E is weak-Freudenthal. In Riesz space terms, this reads as follows.

Definition 0.62
E is called weak-Freudenthal if for every $e \in E^+$, $0 \leq a \leq e$ and $\varepsilon \in (0,\infty)$ there exist disjoint $e_1, \dots, e_n \in E^+$ and scalars $\alpha_1, \dots, \alpha_n \in [0,\infty)$ such that

$$\sum_{i=1}^n e_i = e\,, \text{ and } |a - \sum_{i=1}^n \alpha_i e_i| \leq \varepsilon e\,.$$

In words: if a lies in the ideal generated by e, then we can e-uniformly approximate a with linear combinations of components of e.
To familiarize the reader with the weak-Freudenthal property we mention some less trivial reformulations ([La, 2.3, 2.8]).

Lemma 0.63
The following are equivalent:
(α) *E is weak-Freudenthal.*
(β) *Every principal ideal of E is Riesz isomorphic to a $\| \|_\infty$-dense Riesz subspace of a $C(S)$ with S a zero-dimensional compact Hausdorff space.*
(γ) *If I, J are two ideals of E such that $I + J = E$, then there exists a band B with $B \subset I$, $B^\perp \subset J$ and $B + B^\perp = E$.*
(δ) *For every pair $u, v \in E^+$ there exist disjoint $u_1, v_1 \in E^+$ such that $u_1 \in E_{[u]}$, $v_1 \in E_{[v]}$, and $E_{[u_1+v_1]} = E_{[u+v]}$.*

Further, the weak-Freudenthal property is related to more familiar properties:

Lemma 0.64
i. *Suppose that E is conditionally σ-laterally complete. Then E has the principal projection property.*
ii. *Suppose that E has the principal projection property. Then E has the weak-Freudenthal property.*

$\lhd$ *i:* Let $e \in E^+$. For each $u \in E^+$ there exists a half-disjoint sequence $(u_n)_n$ (see 0.34) such that $\{u_n\}_n$ has the same upper bounds as $\{u \wedge ke : k \in \mathbb{N}\}$. Since E is conditionally σ-laterally complete, $u_e := \sup_n u_n = \sup\{u \wedge ke : k \in \mathbb{N}\}$ exists and it follows from 0.36 that $E_{<e>}$ is a projection band.
ii: Let $0 \le a \le e$ in E^+, and $\varepsilon \in (0, \infty)$. Choose $n \in \mathbb{N}$ such that $n^{-1} \le \varepsilon$ and set

$$e_i := P_{(a-\frac{i-1}{n}e)^+}(a) - P_{(a-\frac{i}{n}e)^+}(a) , \quad (i = 1 \dots n) ,$$

and observe that in a Yosida representation of $E_{[e]}$ that identifies e with $\mathbb{1}$, we have that $P_{(a-\beta e)^+}(a)$ corresponds to $a\mathbb{1}_{\overline{[a>\beta e]}}$ (cf. 0.37)$\rhd$

Examples 0.65
- Suppose E has the principal projection property. Then every principal ideal is Riesz isomorphic to a norm dense Riesz subspace of a $C(S)$ with S basically disconnected (cf 0.37). Since basically disconnected compact Hausdorff spaces are zero-dimensional, it follows that E is weak-Freudenthal.
- The Riesz space of all convergent sequences c is Riesz isomorphic to $C(X)$ with $X = \{0, 1, 2^{-1}, 3^{-1}, \dots\}$ which is zero-dimensional. Therefore, c is weak-Freudenthal. However, c lacks the principal projection property: $u = (1, 0, 2^{-1}, 0, 3^{-1}, 0, \dots) \in c$, but the projection of $(1, 1, 1, \dots)$ onto the band generated by u does not exist in c.

Lemma 0.66
Let E be a weak-Freudenthal Riesz space.
Let $a_1, \dots, a_n \in E^+$ and $\varepsilon > 0$.
Then there exist e_j ($1 \le j \le m$) in E^+ disjoint, and α_{ij} ($1 \le i \le n$, $1 \le j \le m$) in $[0, \infty)$ such that $\sum_{j=1}^m e_j = \sum_{i=1}^n a_i$ (=: e), and

$$| a_i - \sum_{j=1}^{m} \alpha_{ij} e_j | \le \varepsilon e \quad (\text{all } i) .$$

In fact we can choose α_{ij} such that

$$0 \le \bigvee_j (\alpha_{ij} - \varepsilon)^+ e_j \le a_i \le \bigvee_j \alpha_{ij} e_j \quad (\text{all } i) . \qquad (\#)$$

Proof
Let D be the principal ideal generated by $e := \sum_1^n a_i$. By Yoshida's theorem and the fact that E is weak-Freudenthal, we may identify D with a norm-dense Riesz subspace of a $C(S)$ with S zero-dimensional and we may identify e with $\mathbb{1}_S$. By compactness there are $x_1, \dots, x_N \in S$ and clopen $U_1, \dots, U_N$ such that

$$x_j \in U_j \subset [\ |a_1 - a_1(x_j)| < \varepsilon/2\ ] \cap \dots \cap [\ |a_n - a_n(x_j)| < \varepsilon/2\ ]$$

and

$$\bigcup_1^N U_i = X\ .$$

Using 0.31,

$$e_j := \mathbb{1}_{U_j \setminus (U_1 \cup \dots \cup U_{j-1})} \in D\ , \qquad \alpha_{ij} := \max_{U_j} a_i \qquad (1 \le i \le n\ ,\ 1 \le j \le N)\ ,$$

satisfy the requirements. □

#0.3.9 Riesz space completions

Often it is convenient to view E as an order dense Riesz subspace of an enveloping Riesz space F in which one can form suprema of certain subsets of F.
Requiring the existence of suprema of majorized subsets of F induces the Dedekind completion.

Definition 0.67
A Dedekind complete Riesz space F is called a Dedekind completion of E if E can be identified (via a Riesz isomorphism) with an order dense and majorizing Riesz subspace of F.

Such a completion exists, e.g. embed E order densely in a $C^\infty(S)$ (0.11) and take for F the ideal of $C^\infty(S)$ generated by E. All Dedekind completions of E are Riesz isomorphic, which justifies speaking of "the" Dedekind completion of E, denoted by E^δ.

Example 0.68
The Dedekind completion of $C(S)$ is $B(S)$: the Riesz space of all bounded continuous real-valued functions that have dense F_σ-subsets of S as domains, with identification of functions that are equal on a dense subset of S.

Using the Dedekind completion we define

Definition 0.69
The σ-Dedekind completion of E (notation: E^σ) is the smallest σ-Dedekind Riesz subspace of E^δ that contains E (i.e. the intersection of all σ-Dedekind complete Riesz subspaces of E^δ that contain E). E may not lie super-order dense in its σ-Dedekind completion, but if it does we say that E is almost σ-Dedekind complete.

One can express almost σ-Dedekind completeness in less artificial terms: E is almost σ-Dedekind complete if and only if for every decreasing sequence $(u_n)_n$ that is bounded from below, there exists an increasing sequence $(v_n)_n$ such that $u_n - v_n \downarrow 0$.
If S is an uncountable set, then the Riesz space of functions on S that are constant outside a finite set, $c_{00}(S) + \mathbb{R}\mathbb{1}$, has as its Dedekind completion $\ell^\infty(S)$. Its

σ-Dedekind completion consists of all functions on S that are constant outside a countable set, whence it is almost σ-Dedekind complete. $C[0,1]$ is almost σ-Dedekind complete because of its countable sup property.

Corollary 0.70
Every element of $E^{\delta +}$ is a supremum of an upwards directed system in E^+; every element of $E^{\sigma +}$ is the supremum of an increasing sequence in E^+ if E is almost σ-Dedekind complete.

Another kind of Riesz space completion is obtained if we require that F contains E as an order dense Riesz subspace and that disjoint subsets of F^+ have suprema. First some terminology:

Definition 0.71
F is called (σ-)laterally complete if every non-empty (countable) disjoint system in F^+ has a supremum in F.
F is called conditionally (σ-)laterally complete if every majorized non-empty (countable) disjoint system in F^+ has a supremum in F^+.
F is called (σ-)universally complete if it is (σ-)laterally complete as well as (σ-)Dedekind complete.

If S is extremally disconnected, then $C^\infty(S)$ is laterally complete, and therefore universally complete. If S is basically disconnected, then $C^\infty(S)$ is σ-universally complete.

$\triangleleft$We only prove the first. Let $\mathcal{F}$ be a system in $C^\infty(S)^+$ with S extremally disconnected. With g being the pointwise supremum of $\mathcal{F}$, the supremum of $\mathcal{F}$ in the lattice of all continuous functions $S \to [-\infty,\infty]$ is $g^{\dagger}$ (cf. 0.11). We are done if disjointness of $\mathcal{F}$ implies that $[g^{\dagger}=\infty]^\circ = \varnothing$. To see how this comes about, let $U_{\mathcal{F}} := \cup_{f\in\mathcal{F}}[f>0]$ and set $U := U_{\mathcal{F}} \cup (S \setminus U_{\mathcal{F}})^\circ$. Assuming disjointness of $\mathcal{F}$, we see that $g^{\dagger} = f$ on $[f>0]$ $(f \in \mathcal{F})$, whence $[g^{\dagger}=\infty]^\circ \cap [f>0] = \varnothing$ for all $f \in \mathcal{F}$. Also $g^{\dagger} = 0$ on $S \setminus U_{\mathcal{F}}$. Thus, $[g^{\dagger}=\infty]^\circ \cap U = \varnothing$, and so $[g^{\dagger}=\infty]^\circ = \varnothing$ because U is open and dense.$\triangleright$

The (conditionally) (σ-)lateral completion is defined in terms of yet another Riesz space completion: the universal completion.

Definition 0.72
F is called a universal completion of E if E is an order dense Riesz subspace of F and F is universally complete.

As would be expected, there exists a universal completion and all universal completions of E are Riesz isomorphic. More specifically, the universal completion of E, notation E^u, can be represented as a $C^\infty(S)$ (cf. 0.30):

Theorem 0.73 (Maeda-Ogasawara-Vulikh)
Let E be an Archimedean Riesz space.
Then there exists an extremally disconnected compact Hausdorff space S such that E is an order dense Riesz subspace of $C^\infty(S)$, i.e. $C^\infty(S)$ is the universal completion of E.

In terms of the universal completion E^u we can define various types of lateral completions.

Definition 0.74
The (σ-)lateral completion of E, denoted by E^λ ($E^{\sigma\lambda}$), is the intersection of all (σ-)laterally complete Riesz subspaces of E^u that contain E. Since E lies order densely in E^u, E^λ ($E^{\sigma\lambda}$) is the smallest (σ-)laterally complete Riesz subspace of E^u that contains E.
The conditionally (σ-)lateral completion of E, denoted by E^λ_c ($E^{\sigma\lambda}_c$), is the smallest conditionally (σ-)laterally Riesz subspace of E^u that contains E. It is in fact the ideal of E^λ ($E^{\sigma\lambda}$) generated by E.

A more explicit description of the elements of E^λ has been given by Van Dinther ([Di]): represent E^u as $C^\infty(S)$. We say that $f \in C^\infty(S)$ is locally in E at $a \in S$ if and only if there are a neighborhood U_a of a and an $f_a \in E$ such that $f = f_a$ on U_a. Given $f \in C^\infty(S)$, the set of points where f is locally in E, is open. If it is in addition dense, we say that f lies locally in E. Let $C^\infty_{\mathrm{loc}}(S)$ be the collection of all functions in $C^\infty(S)$ that are locally in E. From the definition one checks that $C^\infty_{\mathrm{loc}}(S)$ is a Riesz space. Further, the $C^\infty(S)$-supremum of a disjoint collection in E^+, is in $C^\infty_{\mathrm{loc}}(S)$ (use e.g. the analogon of 0.10 with S extremally disconnected and O open, [LuZa, Thm. 47.1, p. 322]). The converse is not true, but one can show that every positive element of $C^\infty_{\mathrm{loc}}(S)$ is the supremum of -what we will call- a half-disjoint system in E.

Definition 0.75
A system H in E is called half-disjoint if $H = H_1 \cup H_2$ where both H_1 and H_2 are disjoint systems in E.

Of course, a disjoint system is half-disjoint. Every doubleton is half-disjoint.

Lemma 0.76 (Van Dinther)
Every element of $C^\infty_{loc}(S)^+$ is the supremum of a half-disjoint system in E^+.

Proof
The key point is to observe that

$$u, e \in E^+ \Rightarrow u\mathbb{1}_{\overline{[e>0]}} \text{ is the supremum of a half-disjoint sequence in } E^+ .$$

◁ Indeed, by lemma 0.34 there exists under the given circumstances a half-disjoint sequence $(u_n)_n$ in E^+ that has the same upper bounds as $\{u \wedge ke : k \in \mathbb{N}\}$, whence

$$u\mathbb{1}_{\overline{[e>0]}} = P_e^{C^\infty(S)}(u) = C^\infty(S)\text{-}\sup\{u \wedge ke : k \in \mathbb{N}\} = C^\infty(S)\text{-}\sup_n u_n .$$

▷

Let now $f \in C^\infty_{\mathrm{loc}}(S)^+$, and let U_f be the (necessarily) open and dense set of points where f lies locally in E. To establish 0.76, we will find a *disjoint* collection of functions of the form $f_W\mathbb{1}_{\overline{[v>0]}}$ ($W \in \mathcal{W}$, $v \in \mathcal{V}_W$) with $f_W, v \in E^+$ such that

$$f = C^\infty(S)\text{ - }\sup_{W\in\mathcal{W}}\ \sup_{v\in\mathcal{V}_W} f_W\mathbb{1}_{\overline{[v>0]}} .$$

(I): Choice of the collection $\mathcal{W}$ in U_f and disjoint system $(f_W)_{W\in\mathcal{W}}$ in E^+
Using Zorn's lemma we choose a maximal disjoint system $\mathcal{W}$ in $\{W \subset U_f : W \text{ clopen}\}$ that has for each $W \in \mathcal{W}$ an $f_W \in E^+$ with $f = f_W$ on W. By definition of U_f and maximality of $\mathcal{W}$, the union of $\mathcal{W}$ is dense in U_f, which is in its turn dense in S. Therefore, the $f\mathbb{1}_W = f_W\mathbb{1}_W$, $W \in \mathcal{W}$, form a disjoint system in $C^\infty(S)$ having f as supremum.

(II): Choice of the disjoint systems $\mathcal{V}_W$, $W \in \mathcal{W}$, in E^+

Take $W \in \mathcal{W}$. Invoking Zorn's lemma, we choose a maximal disjoint system $\mathcal{V}_W$ in $\{v \in E^+ : v \le \mathbb{1}_W\}$. Noting that E is order dense in $C^\infty(S)$, we see that the maximality implies that $\mathbb{1}_W = \sup_{v \in \mathcal{V}_W} \mathbb{1}_{\overline{[v>0]}}$. □

Recalling that $C^\infty_{\mathrm{loc}}(S)$ is a Riesz space that contains $C^\infty(S)$-suprema of disjoint systems in E^+, we conclude:

Corollary 0.77
Let E be an order dense Riesz subspace of $C^\infty(S)$. Then:

$$f \in C^\infty_{loc}(S)^+ \quad \Longleftrightarrow \quad f \text{ is the supremum of a half-disjoint system in } E^+ .$$

Consequently, $C^\infty_{loc}(S)$ is the smallest laterally complete Riesz subspace of $C^\infty(S)$ that contains E, in other words, $E^\lambda \simeq C^\infty_{loc}(S)$.

Unfortunately, we have not been able to prove an analogon of 0.76 for the σ-lateral completion nor for the conditionally σ-lateral completion.
This situation improves if we require E to have the countable sup property.

Lemma 0.78
Suppose E has the countable sup property. Then every element of the conditionally σ-lateral completion $E_c^{\sigma\lambda}$ is a supremum of a half-disjoint countable system in E^+.

◁Indeed: if $u^{\sigma\lambda} \in E^{\sigma\lambda^+}$, then there exist a half-disjoint system U in E^+ and a v in E^+ such that $\sup U = u^{\sigma\lambda} \le v$. By the countable sup property, U is countable.▷

0.3.10 Relations between the properties

To gain some overview, we summarize and extend the hierarchy of the properties introduced so far ([LuZa, 25.1,p. 137], [AlBu, 2.12,p. 14]).

Theorem 0.79
Denoting by UC universally completeness, by [c](σ)LC [conditionally] (σ) laterally complete, by [A](σ-)DC [almost](σ-)Dedekind complete, by (P)PP the property of having the (principal) projection property, by WF weak Freudenthal, and by Arch Archimedean, we have for E the following implications

$$\begin{tikzcd}
 & DC \arrow[rr] \arrow[dr] & & \sigma DC \arrow[r] \arrow[dr] & A\sigma DC \arrow[dr] & \\
UC \arrow[ur] \arrow[dr] & & PP \arrow[rr] & & PPP \arrow[dr] & Arch \\
 & LC \arrow[rr] \arrow[ur] & & \sigma LC \arrow[r] & c\sigma LC \arrow[u] & WF \arrow[u]
\end{tikzcd}$$

Except for (σ)LC and UC, the properties mentioned above pass on to ideals.
As an application of that, we see that $C(S)$ is Dedekind complete if S is extremally disconnected, because $C(S)$ is a Riesz ideal of $C^\infty(S)$.
We conclude with an elementary but useful property:

Lemma 0.80
Let E_0 be a Dedekind complete and order dense Riesz subspace of E. Then E_0 is an ideal of E.

$\lhd$Indeed, if $u_0 \in E_0^+$, and $e \in E$ with $0 \le e \le u_0$, then by order denseness $e = E\text{-}\sup U$, where $U = \{u \in E_0 \cdot 0 \le u \le e\}$. By the Dedekind completeness of E_0, $E_0\text{-}\sup U$ exists, and by order denseness it equals e.$\rhd$

Remarks 0.81
If E has the weak-Freudenthal property, then every non-zero band contains a non-zero projection band (in which case E is usually said to have sufficiently many projections).

0.3.11 Relations between the order completeness of $C(S)$ and the topological disconnectedness of S

As an illustration of the interplay between the Riesz space structure of $C(S)$ and the topological structure of S, we mention the following.

Lemma 0.82
Let S be a compact Hausdorff space. Then

- *$C(S)$ is σ-Dedekind complete $\iff$ $C(S)$ is conditionally σ-laterally complete $\iff$ $C(S)$ has the principal projection property $\iff$ S is basically disconnected*
- *$C(S)$ is Dedekind complete $\iff$ $C(S)$ is conditionally laterally complete $\iff$ $C(S)$ has the projection property $\iff$ S is extremally disconnected.*
- *$C(S)$ is weak-Freudenthal $\iff$ S is zero-dimensional.*

#0.3.12 The order dual

Using the concept of order boundedness we can single out a special type of linear map. We begin in the general setting of partially ordered vector spaces.

Definition 0.83
Let G, H be partially ordered vector spaces and let Ψ be a map $G \to H$.
Ψ is called order bounded if Ψ maps order bounded subsets of G into order bounded subsets of H. The collection of all linear order bounded maps from G into H is denoted by $\mathcal{L}_b(G,H)$.
Ψ is called positive if $\Psi(x) \ge 0$ in H, whenever $x \ge 0$ in G. With the ordering defined by: $\Psi \ge \Phi$ if and only if $\Psi - \Phi$ is positive, $\mathcal{L}_b(G,H)$ becomes a partially ordered vector space.
In the special case $H = \mathbb{R}$, we call $\mathcal{L}_b(G,H) =: G^\sim$ the order dual of G, and elements of $G^\sim$ are referred to as order bounded linear functionals.

A positive map is order bounded, and so is a linear combination of two positive maps.
We now address the question when is $\mathcal{L}_b(G,H)$ a Riesz space. For this, we need more than that G and H are just Riesz spaces:

Theorem 0.84 ([AlBu, Thm. 3.3, p. 20])
Let E, and F be Riesz spaces with F Dedekind complete.
Then $\mathcal{L}_b(E,F)$ is a Riesz space, and the relevant Riesz space operations are given by

$$\left.\begin{aligned}(\Psi \vee \Phi)(u) &= \sup\{\Psi(v) + \Phi(w) : v + w = u \text{ in } E^+\}\,,\\ (\Psi \wedge \Phi)(u) &= \inf\{\Psi(v) + \Phi(w) : v + w = u \text{ in } E^+\}\,,\\ (\Psi^+)(u) &= \sup\{\Psi(v) : 0 \le v \le u\}\,,\\ (\Psi^-)(u) &= \inf\{\Psi(v) : 0 \le v \le u\}\,,\\ (|\Psi|)\,(u) &= \sup\{\Psi(v) : |v| \le u\}\,,\end{aligned}\right\}\quad (u \in E^+)\,.$$

In particular, $E^\sim$ *is a Riesz space, and*

$$|\Psi(x)| \le |\Psi|(|x|) \quad (x \in E)\,. \tag{1}$$

#0.3.13 Complex Riesz spaces

The complex vector space $L^p_{\mathbb{C}}(S, \mathcal{A}, \mu)$, consisting of all $f : S \to \mathbb{C}$ for which $\mathrm{Re}(f)$, $\mathrm{Im}(f) : S \to \mathbb{R}$ are $\mathcal{A}$-measurable and $\int_S |f|^p \, d\mu < \infty$, can be related to the framework of Riesz spaces as the complexification of the Riesz space $L^p(\mu)$.

Definition 0.85 (Complexification)
Let E be a uniformly complete Riesz space. Then the complexification of E is defined as the $\mathbb{C}$-vector space $E + iE := \{f + ig : f, g \in E\}$ equipped with (the obvious addition and $\mathbb{C}$-scalar multiplication and) the complex absolute value

$|f + ig| := [f^2 + g^2]^{1/2}$ *that is described below.*

Definition 0.86
Let E be a uniformly complete Riesz space. Take $f, g \in E$. By Yoshida's representation theorem, there is a Riesz isomorphism $\widehat{\ }$ from $E_{[|f|+|g|]}$, the principal ideal generated by $|f| + |g|$, onto a $C(S)$ for some compact Hausdorff space S. The function $[\hat{f}^2 + \hat{g}^2]^{1/2}$ is in $C(S)$, and, taking inverse images, we use this element to define $[f^2 + g^2]^{1/2}$ in $E_{[|f|+|g|]}$, whence in E.

Definition 0.87 (Complex Riesz spaces, ideals, bands)
A complex vector space that appears as the complexification of some uniformly complete Riesz space is called a complex Riesz space.
Using the complex absolute value in a complex Riesz space one defines as before (cf. section 0.3.3) the concepts of a complex solid set, and a complex ideal. It turns out that $A \subset E + iE$ is an complex ideal if and only if $A \cap E$ is an ideal in E.
A complex band is defined as a complex ideal B for which $B \cap E$ is a band in E.

Taking $E = L^p(\mu)$, we obtain as complexification $L^p_{\mathbb{C}}(\mu)$ with

$$|f| := [\ |\mathrm{Re}(f)|^2 + |\mathrm{Im}(f)|^2\ ]^{1/2}$$

as complex absolute value. The complex bands in $L^p_{\mathbb{C}}(\mu)$ (for μ σ-finite) again correspond to measurable sets.

0.4 Quasi-normed spaces

Besides its order structure, the metric structure of an L^p-space, which is that of a quasi-normed space, will play a rôle in the characterizations we are aiming at. We start by introducing the concept of a quasi-normed space from the general context of topological vector spaces.

0.4.1 Vector space topologies and pseudonorms

Let (E, τ) be a topological vector space, that is, E is a vector space and τ is a vector space topology: a topology that makes the vector space operations of E continuous. One way to describe τ is to give a base $\mathcal{B}$ of neighborhoods of 0: a base of neighborhoods of an arbitrary $a \in E$ consists of translates of the elements of $\mathcal{B}$. In fact, one can restrict oneself to balanced 0-neighborhoods.

Lemma 0.88 ([Kö, §15.1(3), p. 146] [Ed, 1.8.2(5), p. 57])
A 0-neighborhood U is called balanced if $rU \subset U$ for each scalar r with $|r| \le 1$. Every 0-neighborhood contains a balanced 0-neighborhood (by the continuity of the map $(r, x) \to rx$ at $(0,0)$ with respect to the product topology).

An alternative description of τ is via the τ-continuous pseudonorms.

Definition 0.89
A functional $\rho : E \to [0, \infty)$ is called a pseudonorm if
- *i.* $\lim_{r \to 0} \rho(rx) = 0$ *for all* $x \in E$;
- *ii.* $\rho(rx) \le \rho(x)$ *if* $x \in E$ *and* $|r| \le 1$;
- *iii.* $\rho(x + y) \le \rho(x) + \rho(y)$ *for all* $x, y \in E$ *(subadditivity);*

A collection of pseudonorms $\mathcal{P}$ on E generates a vector space topology τ by setting $x_\alpha \to x$ if and only if for each $\rho \in \mathcal{P}$: $\rho(x - x_\alpha) \to 0$. Equivalently, the sets $[\rho < r]$, with $r \in (0,1)$ and $\rho \in \mathcal{P}$, constitute a 0-neighborhood subbase.

Example 0.90
- Let $(S, \mathcal{A}, \mu)$ be a finite measure space, and let M be the Riesz space of all measurable functions with identification of functions that are equal μ-almost everywhere. Then $f \to \int |f| \wedge \mathbb{1}$ is a pseudonorm generating the topology corresponding to convergence in measure.
- If μ is only locally finite, then for every set of finite measure $A \in \mathcal{A}$ we have a pseudonorm $\rho_A : f \mapsto \int_A |f| \wedge \mathbb{1}$, and the collection of all ρ_A generates a vector space topology.

Conversely, given a vector space topology τ, there exists a collection of pseudonorms that generates τ. In fact, we can take the collection of τ-continuous pseudonorms:

Lemma 0.91
Let τ be a vector space topology and let W be a 0-neighborhood. Then there exists a pseudonorm ρ_W with $[\rho_W < 1] \subset W$ and such that $\rho_W(x_\alpha - x) \to 0$ whenever $x_\alpha \xrightarrow{\tau} x$.

◁Indeed, first take a balanced 0-neighborhood $V \subset W$ and then choose a sequence of *balanced* 0-neighborhoods $V \supset V_1 \supset V_2 \ldots$ such that for all n

$$V_{n+1} + V_{n+1} + V_{n+1} \subset V_n ,$$

(use the continuity of the map $(x, y) \to x + y$ with respect to the product topology). Next define

$$d(x) := \begin{cases} 0 & \text{if } x \in \bigcap_n V_n \\ 2^{-n} & \text{if } x \in V_n \setminus V_{n+1} \\ 1 & \text{if } x \neq V_1 , \end{cases} \qquad (x \in E) ,$$

and finally set $\rho_W(x) := \inf\{\sum_1^n d(x_\iota) : n \in \mathbb{N}, \sum_1^n x_\iota = x\}$. Then ρ_W is a pseudonorm, and $\rho_W(x_\alpha - x) \le d(x_\alpha - x) \le 2^{-n}$, whenever $x_\alpha - x \in V_n$.▷

Special classes of topological vector space topologies are obtained by requiring (the existence of) a special type of 0-neighborhood base or a special type of generating system of pseudonorms. The two classes that will concern us, are that of the locally convex topologies (generated by semi-norms) and that of the locally bounded topologies (generated by quasi-norms).

Definition 0.92 (seminorms and locally convex topologies)
A set A is called convex if

$$\left.\begin{array}{l} x, y \in A, \\ r, s \in [0,\infty), r + s = 1, \end{array}\right\} \quad \Rightarrow \quad rx + sy \in A.$$

A pseudonorm ρ is called a seminorm if it is absolutely homogeneous:

$$\rho(rx) = |r|\rho(x) \quad \text{for all scalars } r \text{ and } x \in E.$$

A vector space topology τ is called locally convex if it admits a 0-neighborhood base consisting of convex sets; equivalently, τ can be generated by seminorms.

Definition 0.93 (quasi-norms and locally bounded topologies)
A set A is called τ-bounded if for every τ-0-neighborhood U there exists a scalar r such that $A \subset rU$.
A quasi-norm is a functional $\|\!\!\!\;\|\;: E \to [0,\infty)$ that is

i. *separating:*
   $\|x\| = 0$ *if and only if* $x = 0$ $(x \in E)$;
ii. *absolutely homogeneous:*
   $\|rx\| = |r|\,\|x\|$ *for all scalars* r, *and* $x \in E$;
iii. *quasi-subadditive:*
   there is a $K \in [1,\infty)$ *with*

$$\|x + y\| \le K \cdot [\,\|x\| + \|y\|\,] \text{ for all } x, y \in E.$$

An quasi-norm is called r-subadditive if for some $r \in (0,1]$:

$$\|x + y\|^r \le \|x\|^r + \|y\|^r \quad (x, y \in E).$$

Thus, the r^{th} power of an r-subadditive quasi-norm is a pseudonorm.
A vector space topology is called locally bounded if it contains a bounded 0-neighborhood. As we will see in 0.98, the latter is equivalent to saying that τ can be generated by a pseudonorm of the form $\|\;\|^r$, where r is in $(0,1]$ and $\|\;\|$ is an r-subadditive quasi-norm.

Examples 0.94

- A norm is a subadditive quasi-norm.
- For $p \in (0,\infty)$, $\|\;\|_p$ is a quasi-norm. In fact, it is a norm for $p \in [1,\infty)$, and a p-subadditive quasi-norm for $p \in (0,1]$ (see 0.1).
- Define $\|\;\| : C[0,1] \to [0,\infty)$ by $\|f\| := \|f\|_1$ if $|f|$ has zeroes, $\|f\| := 2\,\|f\|_1$ otherwise. Let $f_n := nx \wedge \mathbb{1}$ $(n \in \mathbb{N})$. Then $f_n(0) = 0$, $\|\mathbb{1} - f_n\| \to 0$, while $\|f_n\| = 1 - (2n)^{-1} \not\to 2 = \|\mathbb{1}\|$. Consequently, $\|\;\|$ is not continuous, and a fortiori not r-subadditive.
- A locally bounded topology is automatically Hausdorff, for if U is a bounded 0-neighborhood, then the sets rU, $r \in (0,\infty)$, form a 0-neighborhood base.

0.4.2 Quasi-normed spaces versus locally bounded topological vector spaces

Let E be a vector space.

Definition 0.95

If $\|\!\|$ is a quasi-norm on E, then we call the pair $(E, \|\!\|)$ a quasi-normed space.
If $(E, \|\!\|)$ is a quasi-normed space, then $\|\!\|$ generates a vector space topology $\tau_{\|\!\|}$ on E by taking as 0-neighborhood base the sets $[\|\!\| < r]$, $r \in (0,\infty)$, or equivalently by defining $x_\alpha \xrightarrow{\tau} x$ if and only if $\|x_\alpha - x\| \to 0$.
If E is complete with respect to this topology, then we say that $(E, \|\!\|)$ is a quasi-Banach space.

If $(E, \|\!\|)$ is a quasi-normed space, then any of the sets $[\|\!\| < r]$, $r \in (0,\infty)$, is bounded, so $\tau_{\|\!\|}$ is locally bounded.
Conversely, if τ is a locally bounded vector space topology on E, then τ can be generated by a quasi-norm:

Lemma 0.96

Let τ be a Hausdorff vector space topology on E, and let U be a balanced bounded τ-neighborhood of 0.
Then there is a $\kappa \in [2,\infty)$ such that $U + U \subset \kappa U$, and the Minkowski functional (gauge) of U

$$\|x\|_U := \inf\{r \in (0,\infty) : x \in rU\} \quad (r \in (0,\infty)) ,$$

is a quasi-norm with quasi-subadditivity factor κ, and $\|\!\|$ generates τ.

$\lhd$ Since U is a 0-neighborhood, and $\lambda \mapsto \lambda x$ is continuous, we have that $\|\!\|_U$ is finitely valued. Further, $\|\!\|_U$ is separating, since U is bounded and τ is Hausdorff; it is absolutely homogeneous, because U is balanced. Finally, since the vector sum of two bounded sets is bounded, there exists a $\kappa \geq 2$ such that $U + U := \{x + y \ : \ x, y \in U\} \subset \kappa U$, and that implies that $\|\!\|_U$ is quasi-subadditive with factor κ.
Since U is bounded, the sets rU, $r \in (0,\infty)$ constitute a zero neighborhood base. From the latter and

$$\|x\|_U < \rho \quad \Longrightarrow \quad x \in \rho U \quad \Longrightarrow \quad \|x\|_U \leq \rho \quad (x \in E\ ,\ \rho \in (0,\infty)\ )$$

we conclude that $\|\!\|_U$ generates the topology τ. $\rhd$

The most convenient (or least awkward) type of quasi-norm is one that is r-subadditive (for some $r \in (0,1]$). In fact, r-subadditivity already implies quasi-subadditivity:

Lemma 0.97

Let $\|\!\| : E \to [0,\infty)$ be r-subadditive for some $r \in (0,1]$. Then $\|\!\|$ is quasi-subadditive.
In particular, a separating, absolutely homogeneous, r-subadditive functional is automatically a quasi-norm.

$\lhd$ Indeed, by the r-subadditivity of $\|\!\|$ and the inequality 0.1 (on page 4), we have that $\|x + y\| \leq 2^{(1-r)/r}\ \|x\| + \|y\|)$ $(x, y \in E)$. $\rhd$

A pleasant surprise is that a locally bounded vector space topology can not only be generated by a quasi-norm, but even by an r-subadditive quasi-norm.

Lemma 0.98 (Rolewicz)
Let τ be a locally bounded vector space topology, and let U be a balanced bounded 0-neighborhood. Then there is a $\kappa \in [2,\infty)$ such that $U + U \subset \kappa U$.
Let $r \in (0,1]$ be such that $2^{1/r} = \kappa$, and let

$$\Gamma_r(U) := \{\textstyle\sum_1^n \alpha_i x_i \; : \; n \in \mathbb{N}, \alpha_i \text{ scalars}, \sum_1^n |\alpha_i|^r \le 1\}.$$

Then $\Gamma_r(U)$ is a balanced bounded 0-neighborhood that is in addition r-absolutely convex:

$$x, y \in \Gamma_r(U), \; |\alpha|^r + |\beta|^r \le 1 \quad \Longrightarrow \quad \alpha x + \beta y \in \Gamma_r(U).$$

As a result of the latter, the Minkowski functional of $\Gamma_r(U)$ is r-subadditive, so it is (by 0.96) an r-subadditive quasi-norm generating τ.

◁From its definition, $\Gamma_r(U)$ is easily checked to be r-absolutely convex (hence balanced). That $\Gamma_r(U)$ is bounded follows from $U \subset \Gamma_r(U) \subset 2^{1/r}U$, which can be proven by induction.

◁To begin with, the fact that $U + U \subset \kappa U$ implies that $(2^{-1})^{1/r}U + (2^{-1})^{1/r}U \subset U$. The latter can be strengthened to $(2^{-k_1})^{1/r}U + \ldots + (2^{-k_m})^{1/r}U \subset U$ ($k_1, \ldots, k_m \in \mathbb{N}$, $\sum_1^m 2^{-k_i} = 1$) via induction on $k = \max\{k_1, \ldots, k_m\}$. Indeed, since $\sum_1^m 2^{-k_i} = 1$, the number of terms k_i with $k_i = k := \max\{k_1, \ldots, k_m\}$ is even. In view of the fact that $(2^{-k})^{1/r}U + (2^{-k})^{1/r}U = (2^{-(k-1)})^{1/r}((2^{-1})^{1/r}U + (2^{-1})^{1/r}U) \subset (2^{-(k-1)})^{1/r}U$, we therefore obtain that $(2^{-k_1})^{1/r}U + \ldots + (2^{-k_m})^{1/r}U \subset (2^{-l_1})^{1/r}U + \ldots + (2^{-l_M})^{1/r}U$ with $\max\{l_1, \ldots, l_M\} = k - 1$ and $\sum_1^M 2^{-l_j} = 1$.

Next, if $k_1, \ldots, k_n \in \mathbb{N}$ are such that $\sum_1^n 2^{-k_i} \le 1$, we add $k_{n+1}, \ldots, k_m$ so that $\sum_1^m 2^{-k_i} = 1$. Therefore, $(2^{-k_1})^{1/r}U + \ldots + (2^{-k_n})^{1/r}U \subset U$ for all $k_1, \ldots, k_n \in \mathbb{N}$ such that $\sum_1^n 2^{-k_i} \le 1$.

Finally, suppose that $\alpha_1, \ldots, \alpha_n$ are non-vanishing scalars with $\sum_1^n |\alpha_i|^r \le 1$. Then we choose $k_1, \ldots, k_n \in \mathbb{N}$ with $2^{-k_i} \le |\alpha_i|^r \le 2^{-k_i+1}$, whence $\sum_1^n 2^{k_i} \le 1$, which gives that $(2^{-k_1})^{1/r}U + \ldots + (2^{-k_n})^{1/r}U \subset U$. Using the balancedness of U,

$$\begin{aligned}\alpha_1 U + \ldots + \alpha_n U = |\alpha_1|U + \ldots + |\alpha_n|U &\subset (2^{-k_1+1})^{1/r}U + \ldots + (2^{-k_n+1})^{1/r}U \\ &= 2^{1/r}((2^{-k_1})^{1/r}U + \ldots + (2^{-k_n})^{1/r}U) \subset 2^{1/r}U.\end{aligned}$$

Since $\alpha_1, \ldots, \alpha_n$ were arbitrary, $\Gamma_r(U) \subset 2^{1/r}U$.▷

The r-absolute convexity of $\Gamma_r(U)$, finally, implies that the Minkowski-functional of $\Gamma_r(U)$ is r-subadditive.▷

A handy consequence of the above lemma is

Corollary 0.99
Let $(E, \|\!|\;|\!\|)$ be a quasi-Banach space. Suppose $x_1, x_2, \ldots$ is a sequence in E such that the sequence $(\|\!|x_n|\!\|)_n$ allows an exponential estimate, i.e., there exist a $K \in (0,\infty)$ and an $\varepsilon \in (0,1)$ such that for all n: $\|\!|x_n|\!\| \le K\varepsilon^n$.
Then $\|\!|\;|\!\|$-$\lim_N \sum_1^N x_n$ exists.

◁By the above lemma, there exists for some $r \in (0,1)$ an r-subadditive quasi-norm $\|\!|\;|\!\|_{(r)}$ that is equivalent to $\|\!|\;|\!\|$. Then $(\|\!|x_n|\!\|_{(r)})_n$ allows an exponential estimate too, and therefore $\sum_1^\infty \|\!|x_n|\!\|_{(r)}^r < \infty$. The latter, together with the r-subadditivity of $\|\!|\;|\!\|$, implies that the sequence $(\sum_1^N x_n)_N$ is $\|\!|\;|\!\|_{(r)}$-Cauchy, hence $\|\!|\;|\!\|$-Cauchy, and thus convergent.▷

In keeping with the normed-case situation, we define the dual E' of a quasi-normed space $(E, \|\!|\;|\!\|)$ as the collection of all linear functionals $\phi : E \to \mathbb{R}$ for which $\|\!|\phi|\!\|' := \sup\{|\phi(x)| : x \in E, \|\!|x|\!\| \le 1\} < \infty$. Since $(\mathbb{R}, |\;|)$ is a Banach space, $\|\!|\;|\!\|'$ is a norm and $(E', \|\!|\;|\!\|')$ is a Banach space.

0.5 Quasi-normed Riesz Spaces

We now discuss the natural interaction between the metric (quasi-normed space) structure and the ordering (Riesz space) structure in L^p-spaces. The proper context to do that is that of quasi-normed Riesz spaces.

Definition 0.100

Suppose we have a quasi-normed space $(E, \|\ \|)$ and assume that E carries in addition the structure of a Riesz space.

(i) *$\|\ \|$ is called Riesz if only if $|x| \le |y|$ implies $\|x\| \le \|y\|$ $(x, y \in E)$.*

(ii) *If $\|\ \|$ is Riesz, then we refer to $(E, \|\ \|)$ as a quasi-normed Riesz space .*

(iii) *If, in addition, $(E, \|\ \|)$ is a quasi-Banach space, then we call $(E, \|\ \|)$ a quasi-Banach lattice.*

If in the above $\|\ \|$ is actually a norm, then we obtain the familiar concepts of a normed Riesz space and Banach lattice respectively.

Examples 0.101

- The spaces L^p, $p \in (0, \infty)$, are quasi-normed Riesz spaces (in fact quasi-Banach lattices); for $p \in [1, \infty)$, L^p is a Banach lattice.
- The space $C(X)$, where X is a compact Hausdorff space, is a Banach lattice; further, if $D \subset C(X)$ is a $\|\ \|_\infty$-closed Riesz subspace, then D is a Banach lattice too.
- If $\|\ \|$ is a Riesz quasi-norm, then its null-space $N_{\|\ \|} := \{x \in E : \|x\| = 0\}$ is an ideal.

The unit "ball" of a Riesz quasi-norm is solid (by virtue of the Riesz property) and bounded (ensuing from the quasi-norm properties). The converse (and more) is also readily verified (cf. 0.122, p. 39 further on):

Lemma 0.102

Let τ be a locally bounded, locally solid vector space topology in E.
Then E contains a bounded, solid 0-neighborhood U, and and its Minkowski functional is a quasi-norm.

Like normed (Riesz) spaces, r-subadditively quasi-normed Riesz spaces have unique completions (which are quasi-Banach lattices, of course).

#Lemma 0.103 (Completions of r-subadd. quasi-normed Riesz spaces)

Let $r \in (0, 1]$ and let $(E, \|\ \|)$ be a quasi-normed Riesz space with $\|\ \|$ r-subadditive. Then there exists a quasi-Banach lattice $(\hat{E}, \widehat{\|\ \|}_{(r)})$ such that E is $\widehat{\|\ \|}_{(r)}$-dense in $\hat{E}$, $\widehat{\|\ \|}_{(r)}$ is r-subadditive, and such that $\widehat{\|\ \|}_{(r)}$ coincides with $\|\ \|_{(r)}$ on E.

$\triangleleft\, d(x, y) := \|x - y\|_{(r)}^r$ is a metric on E, and we can form the metric space completion $(\hat{E}, \hat{d})$. Note that for every $\hat{x} \in \hat{E}$ there exists a sequence $(x_n)_n$ in E such that $\lim_n \hat{d}(x_n, \hat{x}) = 0$. Further, for every other sequence $(x_n')_n$ with $\lim_n \hat{d}(x_n', \hat{x}) = 0$ we have that $\hat{d}(\hat{x}, 0) = \lim_n d(x_n', 0)$.

Therefore $\widehat{\|\hat{x}\|}_{(r)} := [\hat{d}(\hat{x}, 0)]^{1/r}$ $(\hat{x} \in \hat{E})$ is an r-subadditive quasi-norm (as inheritance of $\|\ \|_{(r)}$) that extends $\|\ \|_{(r)}$; E is $\widehat{\|\ \|}_{(r)}$-dense in $\hat{E}$, and $\hat{E}$ is complete with respect to $\widehat{\|\ \|}_{(r)}$. $\triangleright$

However, quasi-norms need not be r-subadditive (0.94) and this has a repercussion: in general they cannot be "isometrically" embedded in a completion. We do have:

#Corollary 0.104

Let $(E, \|\!\|)$ be a quasi-normed space. By 0.98 there exists an $r \in (0,1]$, and a r-subadditive quasi-norm $\|\!\|_{(r)}$ on E that is equivalent to $\|\!\|$ for some $r \in (0,1]$. Then E is $\|\!\|_{(r)}$-dense in the quasi-Banach lattice $(\hat{E}, \hat{\|\!\|}_{(r)})$ of above.

The topology in a quasi-Banach lattice is entirely determined by the order structure as follows from 0.105 *v.* below.

#Lemma 0.105

Let E be a Riesz space, and suppose $D : E^+ \to [0,\infty)$ has the following properties:

(i) $D(u) = 0 \quad \Longrightarrow \quad u = 0 \qquad (u \in E^+);$

(ii) $\lim_{r\downarrow 0} D(ru) = 0 \qquad (u \in E^+);$

(iii) $D(u+v) \le D(u) + D(v) \quad (u, v \in E^+);$

(iv) $u \le v \Rightarrow D(u) \le D(v) \qquad (u, v \in E^+).$

Then:

i. $D(0) = 0$, $D(nu) \le nD(u)$ $(n \in \mathbb{N}, u \in E^+)$.

ii. $d(x,y) := D(|x-y|)$ *defines a (translation-invariant) metric on E for which the vector space operations are continuous.*

iii. Every relatively uniformly convergent sequence is d-convergent, hence the d-topology is weaker than the relatively uniform topology;

iv. The following are equivalent:

(α) *E is d-complete;*

(β) $\left.\begin{array}{l} x_1, x_2, \ldots \in E^+, \\ \sum_1^\infty D(x_n) < \infty, \end{array}\right\} \quad \Longrightarrow \quad \sup_n\{x_1 + \ldots + x_n\}$ *exists;*

(γ) *If $x_1, x_2, \ldots \in E^+$, $\sum_1^\infty D(x_n) < \infty$ then $s := \sup_n\{x_1 + \ldots + x_n\}$ exists and $x_1 + \ldots + x_n \to s$ relatively uniformly as $n \to \infty$;*

(δ) *If $x_1, x_2, \ldots \in E$, $\sum_1^\infty d(x_n, x_{n+1}) < \infty$ then the sequence $(x_n)_n$ converges relatively uniformly (hence d-converges).*

v. If E is d-complete, then every d-convergent sequence has a relatively uniformly convergent subsequence (having the same limit by iii), hence the d-topology and the relatively uniform topology coincide.

◁ *i*: the first and second claim follow from the second and third property of D respectively;

ii: the continuity of the vector space operations follow from

$$D(|(x+y) - (x_n + y_n)|) \le D(|x - x_n|) + D(|y - y_n|),$$

and

$$\begin{aligned} D(|rx - r_n x_n|) &\le D(|r - r_n||x|) + D(|r_n||x - x_n|) \\ &\le D(|r - r_n||x|) + (\sup_n[|r_n|] + 1)\, D(|x - x_n|), \end{aligned}$$

where the 2nd inequality uses the Riesz property of D and the inequality of *i*.
iii: follows from the second property of D.
iv: $(\alpha) \Rightarrow (\beta)$: Suppose E is d-complete, and take $(x_n)_n$ as described in (β). Then the series $\sum_n x_n$ is d-Cauchy, hence convergent, and its d-limit is its supremum (apply lemma 1.4 (p. 47) to the increasing sequence of the partial sums of the x_i's).
$(\beta) \Rightarrow (\gamma)$: Assume (β) and let $(x_n)_n$ be as described in (γ). By a first application of (β), $s := \sup_n\{x_1 + \ldots + x_n\}$ exists. Now take natural numbers $1 \le k_n \uparrow \infty$ such that $\sum_1^\infty k_n D(x_n) < \infty$.

◁E.g. Choose $N_0 := 1 < N_1 < N_2 < \ldots$ in $\mathbb{N}$ such that $\sum_{N_j}^\infty D(x_n) < 4^{-j}$ $(j \ge 1)$, and set $k_n = 2^k$ if $N_k \le n < N_{k+1}$ $(n = 1, 2, \ldots; k = 0, 1, \ldots)$.▷

Using *i*, we see that $\sum_1^\infty D(k_n x_n) < \infty$ and then a second application of (β) shows that $s^* = \sup_n\{k_1 x_1 + \ldots + k_n x_n\}$ exists. For $2 \le N$ in $\mathbb{N}$:

$$\begin{aligned} 0 \le s - (x_1 + \ldots + x_{N-1}) &= \sup_{n>N}\{x_N + \ldots + x_n\} \\ &\le (k_N)^{-1} \sup_{n>N}\{k_N x_N + \ldots + k_n x_n\} \le (k_N)^{-1} s^* , \end{aligned}$$

hence $x_1 + \ldots x_n \to s$ ($s*$-uniformly).
$(\gamma) \Rightarrow (\delta)$: Take (γ) for granted, and let $(x_n)_n$ be as described in (δ). Applying (γ) to the sequences $y_1 = x_1^+, y_n := (x_{n+1} - x_n)^+$ $(n > 1)$ and $z_1 = x_1^-, z_n := (x_{n+1} - x_n)^-$ $(n > 1)$, we see that $x_n = (y_1 + \ldots + y_n) + (z_1 + \ldots + z_n)$ converges relatively uniformly.
$(\delta) \Rightarrow (\alpha)$: If $(x_k)_k$ is d-Cauchy, then we can select a subsequence $(x_{k_n})_n$ that satisfies $\sum_1^\infty d(x_{k_n}, x_{k_{n+1}}) < \infty$, and will d-converge assuming (δ).▷

Corollary 0.106
Let $p \in (0, \infty)$. Then L^p is $\| \|_p$-complete, hence a quasi-Banach lattice.

◁Indeed, (β) of *0.105* holds for $D = \| \|_p^{\min\{1,p\}}$. As a matter of fact, L^p possesses even a stronger property (see 0.4): *If* $0 \le u_n \uparrow$ *in* L^p *with* $\sup_n \|u_n\|_p < \infty$, *then* $\sup_n u_n \in L^p$.
▷

Later on we will utilize the following continuity properties of quasi-norms, most of which are familiar from the normed Riesz space context.

#Definition 0.107
Let $\rrbracket\,\llbracket$ be a Riesz quasi-norm on E.

(i) We call $\rrbracket\,\llbracket$ weak-Fatou if there exists a $K \in (0,1]$ such that

$$0 \le u_\alpha \uparrow u \text{ in } E \quad \Longrightarrow \quad \sup_\alpha \rrbracket u_\alpha \llbracket \ge K \rrbracket u \llbracket .$$

If we can take $K = 1$, $\rrbracket\,\llbracket$ is called Fatou.

(ii) If we restrict the condition above to sequences instead of nets, we obtain the concepts of a weak-σ-Fatou Riesz quasi-norm and σ-Fatou Riesz quasi-norm respectively.

(iii) We say that $\rrbracket\,\llbracket$ is half-disjoint weak-Fatou if there exists a $K \in (0,1]$ such that $\sup_{u \in U} \rrbracket u \llbracket \ge K \rrbracket u \llbracket$, whenever $U \subset E^+$ is a half-disjoint system in E^+ whose supremum is u. If we can take $K = 1$, we say that $\rrbracket\,\llbracket$ is Fatou.

(See the Remarks 2.49 ad 2.42 (p. 103 for an apology for this definition)

Suppose we have a quasi-normed Riesz space and we know from other sources that its topology is normable (i.e. can be generated by a norm).
Can we then generate its topology by a Riesz norm? The answer is yes:

Lemma 0.108
Let E be a Riesz space and let τ be a vector space topology on E.
Suppose that

(i) *τ contains a solid, bounded 0-neighborhood U ($\rrbracket\llbracket_U$ (see 0.96) is a Riesz quasi-norm)*

(ii) *τ contains a convex, balanced and bounded 0-neighborhood V ($\rrbracket\llbracket_V$ is a norm)*

Then τ can be generated by a Riesz norm.

$\triangleleft$It suffices to come up with a bounded, convex and solid 0-neighborhood. Since U is bounded, there is an $r \in (0, \infty)$ with $U \subset rV$. The convex hull of U (consisting of all $\sum_1^n \alpha_i x_i$ with $x_i \in U$ and $\alpha_i \in [0, \infty)$ such that $\sum_1^n \alpha_i = 1$) is convex, contained in the (convex) bounded set rV, whence bounded, and solid (use the Riesz decomposition property 0.13).$\triangleright$

A quasi-normed Riesz space $(E, \rrbracket\llbracket)$ has two duals: one as quasi-normed space (E'), and one as Riesz space ($E^\sim$). As with normed Riesz spaces, there is a simple relation between the two.

#Lemma 0.109
Let E be a Riesz space and let $\rrbracket\llbracket$ be a quasi-norm on E. Then

i. *E' is an ideal of $E^\sim$;*

ii. *E' is a normed Riesz space;*

iii. *$E' = E^\sim$ if E is a quasi-Banach lattice.*

$\triangleleft$The proof is the standard proof known from the normed Riesz space context ([Za, Thm. 102.3, p. 312], [dJvR, Thm. 10.1, p. 63]).
Let $\phi \in E'$, and $u \in E^+$. For $x \in E$ with $|x| \le u$: $|\phi(x)| \le \rrbracket\phi\llbracket' \rrbracket x\llbracket \le \rrbracket\phi\llbracket' \rrbracket u\llbracket =: M_u$, so $\phi[-u, u] \subset [-M_u, M_u]$ i.e. $E' \subset E^\sim$.
Suppose $\psi \in E^\sim$, $\phi \in E'$ and $|\psi| \le |\phi|$ in $E^\sim$. Then for $|x| \le u$ in E:

$$\begin{aligned} |\psi(x)| &\le |\psi|(|x|) \le |\phi|(|x|) \le |\phi|(u) \le \sup\{|\phi(x)| : |x| \le u\} \\ &\le \sup\{\rrbracket\phi\llbracket' \rrbracket x\llbracket : |x| \le u\} \le \rrbracket\phi\llbracket' \rrbracket u\llbracket \,. \end{aligned}$$

Hence $\psi \in E'$, and $\rrbracket\llbracket'$ is a Riesz norm on E'.
Finally, for $0 \le \phi \in E^\sim \backslash E'$ there exist $x_n \in E$ such that $\rrbracket x_n\llbracket = 2^{-n}$, while $|\phi(x_n)| > 2^n$.
If E were complete however, $u = \sum_n |x_n|$ would exist in E^+ (0.99) and

$$\phi(u) \ge \phi(|x_n|) > 2^n \quad \text{for all } n\,,$$

which yields a contradiction.$\triangleright$

We now focus on L^p-spaces and $C(S)$-spaces from the viewpoint of quasi-normed Riesz spaces.

Definition 0.110

(i) *An abstract L^p-space ($p \in (0,\infty)$) is a quasi-Banach lattice with p-additive quasi-norm (i.e. $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$).*

If $p = 1$, we obtain an abstract L-space.

(ii) *An abstract M-space is a Banach lattice with ∞-additive norm (i.e. $\|u+v\| = \|u\| \vee \|v\|$ whenever $u \wedge v = 0$).*

(iii) *An M-norm is a norm on a Riesz space E that satisfies the M-property i.e. $\|u \vee v\| = \|u\| \vee \|v\|$ $(u, v \in E^+)$.*

For $p = 1$, Kakutani ([Ka]) showed that an abstract L^p-space is isometrically Riesz isomorphic to an $L^1(\mu)$, a result which was generalized to $p \in [1,\infty)$ by Bohnenblust:

Theorem 0.111 (Kakutani and Bohnenblust)
Let $(E, \|\ \|)$ be an abstract L^p-space with $p \in [1,\infty)$. Then there exist a measure space $(S, \mathcal{A}, \mu)$ and a Riesz isomorphism Ω from E onto $L^p(S,\mathcal{A},\mu)$ such that $\|\Omega(x)\|_p = \|x\|$ (all $x \in E$).

$\triangleleft$The characterizations we will obtain in chapter 1 and 2 can all be reduced to the case $p = 1$ of which we will provide a proof below.

By 0.30 there is an extremally disconnected compact Hausdorff space (S, τ) (we mention the topology τ on S explicitly) such that $E \subset C^\infty(S)$. Let B be the Riesz space of Borel functions $S \to \mathbb{R}$ and define for $f, g \in B$:

$$f \overset{*}{=} g \iff f(s) = g(s) \text{ for } \tau\text{-almost all } s \in S \quad .\!\!\iff \quad [f \neq g] \text{ is } \tau\text{-meagre.}$$

(Recall that a set is meagre if it is contained in the countable union G of closed subsets such that $G^\circ = \emptyset$) Likewise we define $f \leq^* g$ etc.

(I): For each $f \in B^+$ there is a (necessarily unique) $f' \in C^\infty(S)^+$ such that $f \overset{*}{=} f'$.

First observe that the collection $\mathcal{A} := \{A \subset S\ .\ \exists U \in \tau \text{ such that } \mathbb{1}_A \overset{*}{=} \mathbb{1}_U\}$ is a σ-algebra

$\triangleleft$E.g. if $A \in \mathcal{A}$, then there is a $U \in \tau$ such that $\mathbb{1}_A \overset{*}{=} \mathbb{1}_U$; because U is open, $\overline{U} \setminus U$ is meagre, so that $\mathbb{1}_{S\setminus A} = \mathbb{1}_S - \mathbb{1}_A \overset{*}{=} \mathbb{1}_S - \mathbb{1}_U \overset{*}{=} \mathbb{1}_S - \mathbb{1}_{\overline{U}} = \mathbb{1}_{(S\setminus U)^\circ}$ i e. $S \setminus A \in \mathcal{A}$).$\triangleright$

Consequently: if A is a Borel-set, then there is an open U such that $\mathbb{1}_A \overset{*}{=} \mathbb{1}_U$, whence $\mathbb{1}_A \overset{*}{=} \mathbb{1}_{\overline{U}} \in C(S)$. Therefore, if f is a Borel-step function, then (i) there is an $f' \in C(S)$ such that $f \overset{*}{=} f'$ and (ii) $\| f' \|_\infty \leq \| f \|_\infty$ because f' assumes only those values $\alpha \in \mathbb{R}$ for which $f^{-1}(\alpha)$ is non-meagre.

Further, if f is a bounded Borel function, then there is a sequence of Borel-step functions $(f_n)_n$ such that $\| f - f_n \|_\infty \to 0$. Because $\| f'_n - f'_m \|_\infty \leq \| f_n - f_m \|_\infty$ (all n, m) we have that $(f'_n)_n$ has a limit $f' \in C(S)$ and $f' \overset{*}{=} f$

Finally, if f is an arbitrary Borel-function, then $\arctan(f)$ is a bounded Borel function $S \to (-\pi, \pi)$ and there is a $g \in C(S)$ such that $|g| \leq \pi\mathbb{1}$ and $g \overset{*}{=} \arctan(f)$. Then $f' := \tan(g) \in C^\infty(S)$ and $f' \overset{*}{=} f$.

(II): Define $J : B^+ \to [0,\infty]$ by

$$J(f) := \sup\{\,\|g\|\ .\ g \in E^+,\ 0 \leq g \leq f'\} = \sup\{\,\|g\| : g \in E^+,\ g \leq^* f\}\ .$$

Then

i. $f_1 \leq^* f_2$ *in* $B^+ \Rightarrow J(f_1) \leq J(f_2)$; $\quad f_1 \overset{*}{=} f_2$ *in* $B^+ \Rightarrow J(f_1) = J(f_2)$;

ii. $J(\alpha f) = \alpha \ \ J(f)$ *if* $\alpha \in [0,\infty)$;

iii. $J = \| \ \|$ *on* E^+*, so* J *is additive on* E^+*;*

iv. J *has the* σ*-Levi property on* E*:*

$$\left.\begin{array}{l} 0 \le g_n \uparrow \text{ in } E^+, \\ \sup_n J(g_n) < \infty \end{array}\right\} \Rightarrow \left\{\begin{array}{l} g := C^\infty(S)\text{-}\sup_n g_n \in E^+, \\ J(g) = \sup_n J(g_n). \end{array}\right.$$

For *iv* use 0.105 *iv.* $(\alpha) \Rightarrow (\gamma)$ with $x_n = g_n - g_{n-1}$ and *iii* above

(III): Let $f \in B^+$ *with* $J(f) < \infty$*. Then* $f' \in E^+$*.*

Take $0 \le g_n \uparrow$ in E^+, $g_n \le f'$, such that $J(g_n) \ge J(f) - n^{-1}$. By *iv* above, there is a $g \in E^+$ with $g \le f'$ and $\|g\| = J(f)$. *If* $g < f'$, then there would be an $j \in E^+ \setminus \{0\}$ (E is order dense in $C^\infty(S)$) with $g + j \le f'$. Applying the additivity of $\| \ \|$ would then result in $J(f) = \|g\| < \|g\| + \|j\| = \|g + j\| = J(g + j) \le J(f)$. *Contradiction* Therefore, $f' = g \in E^+$.

(IV): Let $f_1, f_2 \in B^+$*. Then* $J(f_1 + f_2) = J(f_1) + J(f_2)$*.*

We only prove $\le$: if $g \in E^+$ with $g \le (f_1 + f_2)' = f_1' + f_2'$, then $g = g_1 + g_2$ with $0 \le g_i \le f_i'$ (Riesz decomposition property), and $J(g_i) \le J(g) = \|g\| < \infty$ so $g_i \in E^+$ (by *(III)*) and $\|g_1\| + \|g_2\| = \|g\|$.

(V): J *satisfies Levi's monotone convergence theorem on* B^+ *i.e.*

$$0 \le f_n \uparrow f \text{ pointwise in } B^+ \quad \Longrightarrow \quad J(f_n) \uparrow J(f).$$

We only consider the non-trivial case, i.e. $\sup_n J(f_n) < \infty$. In that case, the sequence $(f_n')_n$ is in E^+ (by *(III)*), and it has a supremum in E^+ satisfying $J(\sup_n f_n') = \sup_n J(f_n')$ (by *(II) iv*). Actually, if we denote the pointwise supremum of $(f_n')_n$ by h, then by the remarks between 0.12 (p. 8) and its proof we have that $h^\dagger = C^\infty(S)\text{-}\sup_n f_n'$ (which equals E-$\sup_n f_n'$), and $h^\dagger \overset{*}{=} h$ (because h is upper semi-continuous).

Since $f_n \overset{*}{=} f_n'$, it follows for the corresponding pointwise suprema that $f \overset{*}{=} h$ Therefore, $J(f) = J(h) = J(h^\dagger) = J(E\text{-}\sup f_n') = \sup_n J(f_n') = \sup_n J(f_n)$.

(V): Define $\mu(A) := J(\mathbb{1}_A) \in [0, \infty]$ *(*A *a Borel set in* S*)*

Then μ *is a measure on* $(S, \mathrm{Borel}(S))$ *(use (IV) and (V) above) such that*

i. $\int f \, d\mu = J(f)$ $(f \in B^+)$*, so* $\int |f| d\mu < \infty \iff f' \in E$*;*

ii. *the map* $L^1(\mu) \to E$ *that sends the* μ*-equivalence class of a* μ*-integrable* f *to* $f' \in E$ *is a surjective Riesz isomorphism that preserves the respective norms.*

Indeed, *i* holds if f is a Borel-step function (by (IV) and (II)*ii*), and also for arbitrary $f \in B^+$, since Levi's monotone convergence theorem holds for both J and $\int d\mu$.▷

Shortly after their L^p-characterization, Kakutani and Bohnenblust showed [BoKa] that an abstract M-space is isometrically Riesz isomorphic to (a closed Riesz subspace of) a $C(S)$, with S compact Hausdorff.

Theorem 0.112 (Kakutani and Bohnenblust)

Let $(E, \| \ \|)$ *be an abstract* M*-space. Then there exist a compact Hausdorff space* S *and a Riesz isomorphism* Ω *from* E *onto a* $\| \ \|_\infty$*-closed Riesz subspace of* $C(S)$ *such that* $\| \Omega(x) \|_\infty = \|x\|$ *(all* $x \in E$*).*

In particular, this result shows indirectly that an ∞-additive norm is an M-norm (a fact that was proven directly, i.e. without representation, by Bernau in [Be]). In chapter 2 we will spend some theorems on the question whether the same holds if we replace norms by quasi-norms.

Bohnenblust ([Bo]) combined the two characterizations above to

Theorem 0.113 (Bohnenblust)

Let $(E, \|\ \|)$ be a Banach lattice of dimension at least three such that

$$\|x+y\| = \|x'+y'\| \quad \textit{whenever} \quad \begin{cases} \|x\| = \|x'\|\,,\ \|y\| = \|y'\|\,, \textit{ and} \\ x \wedge y = x' \wedge y' = 0\,. \end{cases}$$

Then:

$(E, \|\ \|)$ is either isometrically Riesz isomorphic to an $L^p(\mu)$ for some $p \in [1,\infty)$ or to a closed Riesz subspace of a $C(S)$ for some compact Hausdorff space S.

The key result here is the following lemma ([AlBu, Lm. 10.17, 10.18, p. 76]):

Lemma 0.114 (Bohnenblust)

Let $\|\ \|$ be a Riesz norm on a Riesz space E satisfying $\|x+y\| = \|x'+y'\|$ whenever $\|x\| = \|x'\|$, $\|y\| = \|y'\|$, and $x \wedge y = x' \wedge y' = 0$ i.e. there is a function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|x+y\| = f(\|x\|\,,\|y\|)$ whenever $x \wedge y = 0$.
If E has at least dimension three, then f is of the form $f(s,t) = (s^p + t^p)^{1/p}$ for some $p \in [1,\infty)$ or $f(s,t) = \max\{s,t\}$.

A glance at the proof, however (e.g. in [AlBu, Lm. 10.17]), shows that if we drop the condition that $\|\ \|$ is subadditive, then f could equally well be of the form $f(s,t) = (s^p + t^p)^{1/p}$ for some $p \in (0,1)$.
Later other isometric characterizations were derived, among which we note

#Theorem 0.115 (Ando [An69])

Let $(E, \|\ \|)$ be a Banach lattice of dimension at least three.
Suppose that for every closed Riesz subspace F there exists a projection $P : E \to F$ such that $\|P(x)\| \le \|x\|$ (P is contractive) and $P(u) \ge 0$ for $u \ge 0$ (P is positive).
Then:
$(E, \|\ \|)$ is either isometrically Riesz isomorphic to an $L^p(\mu)$ for some $p \in [1,\infty)$ or to a $c_0(S)$ for some discrete set S.

Homeomorphic versions were obtained too, among which we note those of Tzafriri ([LiTz2, Thm. 1.b.12, p. 22])

Theorem 0.116 (Tzafriri)

Let $(E, \|\ \|)$ be a Banach lattice such that $\|x_\alpha\| \downarrow 0$ whenever $x_\alpha \downarrow 0$ (i.e. $(E, \|\ \|)$ is order continuous).
Then the following are equivalent:

(α) *There exists a Riesz isomorphism Ω from E onto either an L^p-space (for some $p \in [1,\infty)$) or onto $c_0(S)$ (for some discrete S) that is at the same time a homeomorphism.*

(β) *There exists a function $F : \mathbb{R}^{\mathbb{N}} \to [0,\infty)$ and a constant $K \in [1,\infty)$ so that*

$$K^{-1} \cdot F(\|x_1\|\,, \|x_2\|\,, \ldots) \le \left\|\textstyle\sum_1^\infty x_n\right\| \le K \cdot F(\|x_1\|\,, \|x_2\|\,, \ldots)$$

for all sequences $(x_n)_1^\infty$ in E such that $\sum_1^\infty x_n$ converges.

Both theorems above eventually rely on the Kakutani-Bohnenblust's results.

Finally, we mention that, as normed Riesz spaces, L-spaces and M-spaces are each other duals (e.g. [AlBu, Thm. 10.15, p.74]).

#Theorem 0.117
Let $(E, \| \|)$ be a normed Riesz space.
If E is an L-space, then E' is an M-space;
if E is an M-space, then E' is an L-space.

Non-convex quasi-normed Riesz spaces may not have a dual at all: Day showed that $(L^p[0,1])' = \{0\}$ if $p \in (0,1)$ ([Ko, 15.9(9) p. 158]).

0.6 Locally solid Riesz spaces

Ascending one level of abstraction by leaving the metric structure for the topological, we can view L^p-spaces and M-spaces as so-called locally solid Riesz spaces.

Definition 0.118
Let E be a Riesz space. A locally solid topology on E is a vector space topology τ on E that admits a base of solid 0-neighborhoods. In this case (E, τ) is referred to as a locally solid Riesz space.

Example 0.119
A Riesz quasi-norm generates a Hausdorff locally solid topology. In particular, L^p-spaces and M-spaces are locally solid Riesz spaces.

We can express the local solidness of a topology otherwise.

Lemma 0.120
Let τ be locally solid Riesz space. Then for every τ-zero neighborhood U there exists a τ-continuous Riesz pseudonorm ρ such that $[\rho < 1] \subset U$, i.e. τ is generated by the collection of τ-continuous Riesz pseudonorms.

(The Riesz property for pseudonorms has not yet formally been introduced, but it is self-explanatory (0.100)).

Examples 0.121
- A Riesz quasi-norm is a particular sort of Riesz pseudonorm;
- the pseudonorm $f \mapsto \int |f| \wedge \mathbb{1}$ on M (see 0.90) is Riesz.

The topology generated by a Riesz quasi-norm is both locally bounded and locally solid. In fact:

Lemma 0.122
A quasi-normed Riesz space is a locally bounded, locally solid Riesz space. Conversely, every locally bounded, locally solid Riesz space can be generated by an r-subadditive Riesz quasi-norm (for some $r \in (0,1]$).

$\triangleleft$ Indeed, let τ be a locally solid and locally bounded topology. By first selecting a bounded 0-neighborhood and then a solid (so balanced) neighborhood therein, we obtain a solid and bounded 0-neighborhood U. By the boundedness of U, there is an $r \in (0,1]$ such that $U + U \subset 2^{1/r}U$. As in lemma 0.98, the Minkowski functional $\| \|_{\Gamma_r(U)}$ of $\Gamma_r(U)$ is an r-subadditive quasi-norm. Further, since $\Gamma_r(U)$ inherits the solidness of U (use the Riesz decomposition property), $\| \|_{\Gamma_r(U)}$ is Riesz as well. $\triangleright$

A map Φ from a topological vector space (E_1, τ_1) into another topological vector space (E_2, τ_2) is called uniformly continuous if for every τ_2-neighborhood V_2 of 0

in E_2, there exists a τ_1-neighborhood V_1 of 0 in E_1 such that $\Phi(x) - \Phi(y) \in V_2$ as soon as $x - y \in V_1$.
From Birkhoff's inequalities we see that the lattice operations $\vee, \wedge : E \times E \to E$ are uniformly continuous in a locally solid Riesz space. As a result also the derived operations $^+, ^-, |\ | : E \to E$ are uniformly continuous. The converse is also true ([AlBu, Thm. 5.2, p. 34]).

Lemma 0.123
Let E be a Riesz space and let τ be a vector space topology on E. Then τ is locally solid if and only if one of the lattice operations is uniformly continuous. (In the latter case, all lattice operations are uniformly continuous).

As any Hausdorff topological vector space, a Hausdorff locally solid Riesz space E has a topological vector space completion $\hat{E}$ (cf. [Kö, §15.3, p. 148]). We recall:

Definition 0.124
Let (E, τ) be a Hausdorff topological vector space.
A net $(x_\alpha)_\alpha$ in E is called τ-Cauchy net if for every τ-0-neighborhood U there exists an α_U with $x_\alpha - x_\beta \in U$ whenever $\alpha \succ \alpha_U$ and $\beta \succ \alpha_U$.
E is called topologically complete if every τ-Cauchy net in E, τ-converges to some element of E.
A topological vector space $(\hat{E}, \hat{\tau})$ is called the topological completion of (E, τ) if $\hat{\tau}|_E = \tau$, E is $\hat{\tau}$-dense, in $\hat{E}$ and $\hat{E}$ is $\hat{\tau}$-complete.

If E is a Hausdorff locally solid topology, then the lattice operations are uniformly continuous, and therefore they extend to $\hat{E}$ rendering it a Riesz space as well ([Fr, §22F, p. 41], [AlBu, 4.5, p. 30]). As a result:

#Corollary 0.125 (Completion of a Hausdorff locally solid Riesz space)
Let (E, τ) be a Hausdorff locally solid Riesz space and let $(\hat{E}, \hat{\tau})$ be its topological vector space completion.
Then $\hat{E}$ has a natural Riesz space structure, and $\hat{\tau}$ is locally solid.
Moreover, if $\mathcal{U}$ is a τ-0-neighborhood base in E, then the $\hat{\tau}$-closures of the sets U, $U \in \mathcal{U}$, form a $\hat{\tau}$-zero neighborhood base in $\hat{E}$.
In particular, the Hausdorff locally solid Riesz space completion of a metrizable topological Riesz space is itself metrizable.

We collect some properties of (Hausdorff) locally solid Riesz spaces, most of which are direct consequences of the definitions ([AlBu, Thm. 5.4, 5.6, p. 35]).

Lemma 0.126
Let (E, τ) be a locally solid Riesz space.

i. *Relative uniform convergence implies topological convergence: if $x_n \to x$ e-uniformly for some $e \in E$, then $x_n \xrightarrow{\tau} x$;*

ii. *order bounded sets are topologically bounded;*

iii. *if τ is Hausdorff: $v_\alpha \uparrow$, $v_\alpha \xrightarrow{\tau} v$ implies $v = \sup_\alpha v_\alpha$; likewise: $v_\alpha \downarrow$, $v_\alpha \xrightarrow{\tau} v$ implies $v = \inf_\alpha v_\alpha$ (cf. lemma 1.4).*

In the last two paragraphs, we will introduce locally solid topologies with additional order continuity properties.

#0.6.1 Fatou topologies

#Definition 0.127

A locally solid topology on E is said to have the (σ)-Fatou property (or to be a (σ)-Fatou topology) if it admits a neighborhood base of (σ-)order closed sets. In this case (E, τ) is called a (σ-)Fatou Riesz space.

A Fatou topology allows a convenient characterization in terms of so-called Fatou pseudonorms.

Definition 0.128

A Riesz pseudonorm ρ is called Fatou if $\rho(u) = \sup_\alpha \rho(u_\alpha)$ whenever $0 \le u_\alpha \uparrow u$.
A Riesz pseudonorm that is Fatou is often called a Fatou pseudonorm.
Similarly, if we replace nets by sequences in the above, we obtain the concept of a σ-Fatou pseudonorm.

Since the unit "ball" of a (σ-)Fatou pseudonorm is solid and (σ-)order closed, a collection of (σ-)Fatou pseudonorms generates a (σ-)Fatou topology. On the other hand, Fremlin showed for the Fatou case (and the proof for the σ-Fatou case is almost verbatim the same [Fr, 23B, p.44]):

Lemma 0.129 (Fremlin)

Let U be a neighborhood of 0 in a (σ-)Fatou topology τ.
Then there exists a τ-continuous (σ-)Fatou pseudonorm ρ such that $[\rho < 1] \subset U$.

Conclusion: a (σ-)Fatou topology is generated by its continuous (σ-)Fatou pseudonorms, and has a neighborhood base of 0 consisting of solid and (σ-)order closed sets.

Example 0.130

- $C(X)$ with the $\| \; \|_\infty$-topology is Fatou.
- the norm $\|x\| := \sup_n |x(n)| + \lim_n |x(n)|$ $(x \in c)$ is not Fatou, but the topology it generates is (because $\| \; \|_\infty \le \| \; \| \le 2\| \; \|_\infty$).
- An (σ-)order continuous Riesz pseudonorm is (σ-)Fatou.

We mention some features of Fatou pseudonorms ([Fr, 23C-F]):

Lemma 0.131

Let ρ be a Fatou pseudonorm on E. Then:

i. *$x_\alpha \uparrow x \Rightarrow \rho(x) \le \sup_\alpha \rho(x_\alpha)$;*
ii. *$N_\rho := \{x \in E : \rho(x) = 0\}$ is an order closed ideal, (hence band);*
iii. *if $(x_\alpha)_\alpha$ is a net in E, then $\mathrm{Lim}_\rho(x_\alpha) := \{x \in E : \lim_\alpha \rho(x - x_\alpha) = 0\}$ is an order closed lattice;*
iv. *if $(x_n)_n$ is a sequence in E such that*

   (i) $\rho(x_{n+1} - x_n) \le 2^{-n}$ (all n),

   (ii) $\inf_{k \ge n} x_n$ exists (all n),

   (iii) $\sup_n \inf_{k \ge n} x_n$ exists,

   then $\rho(x_n - \sup_m \inf_{\iota \ge m} x_\iota) \le 2^{-n+1}$;

Proof

i: If $x_\alpha \uparrow x$, then $0 \le x_\alpha{}^+ \uparrow x^+$, and $x^- \le x_\alpha{}^-$ (for all α).
Hence $x_\alpha{}^+ + x^- \le x_\alpha{}^+ + x_\alpha{}^- = |x_\alpha|$, while $0 \le x_\alpha{}^+ + x^- \uparrow x^+ + x^- = |x|$, so that

$$\rho(x) = \rho(|x|) = \sup_\alpha \rho(x_\alpha{}^+ + x^-) \le \sup_\alpha \rho(|x_\alpha|) = \sup_\alpha \rho(x_\alpha) .$$

ii: If $x_\alpha \in N_\rho$, $x \in E$, and $|x - x_\alpha| \le p_\alpha \downarrow 0$, then $|x_\alpha| \ge |x| - p_\alpha \uparrow |x|$, so that by i: $\rho(x) \le \sup_\alpha \rho(|x| - p_\alpha) \le \sup_\alpha \rho(|x_\alpha|) = 0$.
iii: If $x \in \mathrm{Lim}_\rho(x_\alpha)$, then $\mathrm{Lim}_\rho(x_\alpha) = x + N_\rho$, so it is an order closed lattice.
iv: Let $(x_n)_n$ be as in the premise. Then

$$\begin{aligned}\rho(\sup_m \inf_{\iota\ge m} x_\iota - x_n) &= \rho(\sup_{m\ge n}[\inf_{\iota\ge m} x_\iota - x_n])\\ &\overset{(\mathrm{i})}{\le} \sup_{m\ge n} \rho(\inf_{\iota\ge m} x_\iota - x_n) = \sup_{m\ge n} \rho(\sup_{k\ge m}[\inf_{k\ge\iota\ge m} x_\iota - x_n])\\ &\overset{(\mathrm{i})}{\le} \sup_{m\ge n} \sup_{k\ge m} \rho(x_n - \inf_{k\ge\iota\ge m} x_\iota) .\end{aligned}$$

The result now follows from the observation that for $k \ge m \ge n$,

$$\begin{aligned}x_n - \inf_{k\ge\iota\ge m} x_\iota &= \textstyle\sum_{\iota=n}^{m-1}(x_\iota - x_{\iota+1}) + \sum_{\iota=m}^{k-1}(\inf_{\iota\ge j\ge m} x_j - \inf_{\iota+1\ge j\ge m} x_j)\\ &= \textstyle\sum_{\iota=n}^{m-1}(x_\iota - x_{\iota+1}) + \sum_{\iota=m}^{k-1}(\inf_{\iota\ge j\ge m} x_j - x_{\iota+1})^+\\ &\le \textstyle\sum_{\iota=n}^{m-1}(x_\iota - x_{\iota+1}) + \sum_{\iota=m}^{k-1}(x_\iota - x_{\iota+1})^+\end{aligned}$$

whence $\rho(x_n - \inf_{k\ge\iota\ge m} x_\iota) \le \rho(\sum_{\iota=n}^{k-1} |x_\iota - x_{\iota+1}|) \le 2^{-n+1}$. □

Under appropriate conditions, a Cauchy net in a Fatou Riesz space converges and its limit can be constructed out of the ordering by means of liminf-like constructions.

Definition 0.132 ($\mathcal{I}$ncreasing and $\mathcal{D}$ecreasing envelopes)
If C is a subset of a Riesz space F, we denote

$$\mathcal{I}C := \{x \in F : \exists X \subset C, X \uparrow x\} \quad \text{and} \quad \mathcal{D}C := \{x \in F : \exists X \subset C, X \downarrow x\} .$$

Lemma 0.133 ([Fr, 23G-H])
Let (F, τ) be a locally solid Riesz space and let $C \subset F$ such that

i. *F is Dedekind complete,*
ii. *τ is a Hausdorff Fatou topology on F,*
iii. *C is order bounded,*
iv. *C is closed under the operation $\wedge$.*

Suppose $(u_\alpha)_{\alpha\in\mathbb{A}}$ is a τ-Cauchy net in C.
Then there is a $u \in \mathcal{IDID}C$ such that $u_\alpha \xrightarrow{\tau} u$.

Proof of lemma 0.133
Let $\mathcal{P}$ be the collection of all τ-continuous Fatou pseudonorms on F. Setting for $\rho \in \mathcal{P}$,

$$L(\rho) := \mathrm{Lim}_\rho(u_\alpha) \cap \mathcal{ID}C = \{u \in \mathcal{ID}C : \lim_\alpha \rho(u_\alpha - u) = 0\} ,$$

we show in four steps that $\mathrm{Lim}_\rho(u_\alpha) \cap \mathcal{ID}C \ne \emptyset$ (for each $\rho \in \mathcal{P}$) and that $\bigcap_{\rho\in\mathcal{P}} \mathrm{Lim}_\rho(u_\alpha) \cap \mathcal{IDID}C \ne \emptyset$.
(I): Let $\rho \in \mathcal{P}$. Then $L(\rho) \ne \emptyset$.

Using the Cauchy property of $(u_\alpha)_\alpha$ we choose $\alpha_1 \prec \alpha_2 \prec \alpha_3 \prec \dots$ such that $\rho(u_{\alpha_{n+1}} - u_{\alpha_n}) \le 2^{-n}$. Then by 0.131 iv, $\sup_{m\in\mathbb{N}} \inf_{n\ge m} u_{\alpha_n} \in \mathrm{Lim}_\rho(u_\alpha)$. Moreover, since C is closed under taking finite infima,

$$\inf_{n\ge m} u_{\alpha_n} = \inf_{k\ge m}(\inf_{k\ge n\ge m} u_{\alpha_n}) ,$$

so that $\inf_{n\ge m} u_{\alpha_n} \in \mathcal{D}C$ all n, and $\sup_m \inf_{n\ge m} u_{\alpha_n} \in \mathcal{I}\mathcal{D}C$.

Observing that the operations $\mathcal{I}$ and $\mathcal{D}$ preserve closedness under $\wedge$ (by the infinite distributivity 0.13), we have that $\mathcal{I}\mathcal{D}C$ inherits the closedness under $\wedge$ from C.

(II): Let $\rho \in \mathcal{P}$. Then $u(\rho) := \inf L(\rho) \in \mathrm{Lim}_\rho(u_\alpha) \cap \mathcal{D}\mathcal{I}\mathcal{D}C$.
By 0.131 iii, $\inf L(\rho) \in \mathrm{Lim}_\rho(u_\alpha)$, and since $L(\rho)$ is contained in $\mathcal{I}\mathcal{D}C$ which is closed under $\wedge$, $\inf L(\rho) \in \mathcal{D}\mathcal{I}\mathcal{D}C$.

With the pointwise ordering ($\rho_1 \le \rho_2$ if and only if $\rho_1(x) \le \rho_2(x)$ for all $x \in F$), $\mathcal{P}$ becomes a directed set.

(III): $(u(\rho))_{\rho\in\mathcal{P}}$ is an increasing net in $\mathcal{D}\mathcal{I}\mathcal{D}C$.
Take $\rho_1, \rho_2 \in \mathcal{P}$. Then $\rho_1 + \rho_2 \in \mathcal{P}$ and for $v \in L(\rho_1 + \rho_2)$ we have $v \in \mathcal{I}\mathcal{D}C$ and $\rho_i(v - u_\alpha) \le (\rho_1 + \rho_2)(v - u_\alpha) \to 0$ i.e. $v \in \mathrm{Lim}_{\rho_i}(u_\alpha)$ $(i = 1, 2)$.
Therefore, $L(\rho_1 + \rho_2) \subset L(\rho_i)$ $(i = 1, 2)$ and so $u(\rho_1 + \rho_2) \ge u(\rho_1)$ and $\ge u(\rho_2)$.

(IV): Let $u := \sup_\rho u(\rho) \in \mathcal{I}\mathcal{D}\mathcal{I}\mathcal{D}C$. Then $u \in \mathrm{Lim}_\rho(u_\alpha)$ for all $\rho \in \mathcal{P}$.
Take $\rho_0 \in \mathcal{P}$ and set $v(\rho) := u(\rho_0 + \rho)$, $\rho \in \mathcal{P}$. Then $(v(\rho))_\rho$ is a subnet of $(u(\rho))_\rho$ that is cofinal. Hence, $u = \sup_\rho v(\rho)$. From the latter we also see that $u \in \mathrm{Lim}_{\rho_0}(u_\alpha)$, since for all ρ, $v(\rho) \in \mathrm{Lim}_{\rho_0+\rho}(u_\alpha) \subset \mathrm{Lim}_{\rho_0}(u_\alpha)$ and $\mathrm{Lim}_{\rho_0}(u_\alpha)$ is order closed. □

#0.6.2 Lebesgue topologies

Definition 0.134
A locally solid topology τ is said to have the Lebesgue property if $u_\alpha \downarrow 0$ implies $u_\alpha \xrightarrow{\tau} 0$. A locally solid topology with the Lebesgue property is shortly referred to as a Lebesgue topology.
A related but weaker condition is the pre-Lebesgue property: A locally solid topology τ is said to have the pre-Lebesgue property if $(u_n)_n$ is Cauchy whenever $0 \le u_n \uparrow\le v$. A locally solid topology with the pre-Lebesgue property is called a pre-Lebesgue topology.

The justification for the term “pre-Lebesgue” is (see [AlBu, Thm. 10.5, p. 67]):

Lemma 0.135
Let (E, τ) be a locally solid Riesz space. Then E is pre-Lebesgue if and only if its topological completion $\hat{E}$ is Lebesgue.

For the pre-Lebesgue property there exists a convenient criterion (the proof of which can be found in [AlBu, Thm. 10.1, p. 64]):

Lemma 0.136
E is pre-Lebesgue if and only if every disjoint majorized sequence in E^+ converges to 0.

Examples 0.137

- The $\| \|_p$-topology is pre-Lebesgue (by the p-additivity of $\| \|_p$ and the above criterion), and in fact Lebesgue (by its completeness).
- The Riesz subspace $C[0,1]$ of $L^p[0,1]$, equipped with the restriction of the $\| \|_p$-topology, is pre-Lebesgue, but not Lebesgue.

Lebesgue topologies are Fatou. More precisely ([AlBu, Thm. 11.6, p. 81]):

Lemma 0.138
The Lebesgue property is equivalent to the combination of the pre-Lebesgue property and the Fatou property.

Example 0.139
ℓ^∞ with $\| \|_\infty$-topology is Fatou, but not Lebesgue.

A nice feature of Lebesgue property is that topological completeness implies order completeness ([AlBu, 17.10 (p. 119), 10.3 (p. 66)]):

Lemma 0.140
(Metrizable) complete Lebesgue spaces are (super) Dedekind complete.

0.6.3 (σ-)Levi Topologies

Definition 0.141
A locally solid topology is said to have the σ-Levi property (or is called σ-Levi) if

$$\left.\begin{array}{l} 0 \le u_n \uparrow, \\ \{u_n\}_n \ \tau\text{-bounded}, \end{array}\right\} \quad \Longrightarrow \quad \sup_n u_n \ \text{exists in } E\,.$$

If we replace sequences by nets, we obtain the Levi-property.

By lemma 0.4, L^p-spaces are (σ-)Levi.

Chapter 1

Riesz isometric characterizations

1.1 The Kakutani-Bohnenblust characterization

We begin by representing the Kakutani-Bohnenblust characterization (the K.-B. characterization for short) of abstract L^p-spaces in a spelled-out version.

Theorem 1.1 (The Kakutani-Bohnenblust characterization of L^p)
Let E be a Riesz space, let $p \in [1, \infty)$, and let $\| \ \|$ be a p-additive Riesz norm on E, that is, $\| \ \|$ is

i. *separating:* $\|x\| = 0$ *if and only if* $x = 0$ $(x \in E)$;

ii. *absolutely homogeneous:* $\|rx\| = |r| \, \|x\|$ $(x \in E, r \in \mathbb{R})$;

iii. *subadditive:* $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in E)$;

iv. *Riesz:* $|x| \leq |y|$ *implies* $\|x\| \leq \|y\|$ $(x, y \in E)$, *and*

v. *p-additive:* $x \wedge y = 0$ *implies* $\|x + y\|^p = \|x\|^p + \|y\|^p$ $(x, y \in E)$.

Suppose E is topologically complete with respect to $\| \ \|$.
Then there exists a measure space $(S, \mathcal{A}, \mu)$ and a surjective Riesz isomorphism $\Omega : E \to L^p(\mu)$ such that $\|\Omega(x)\|_p = \|x\|$ $(x \in E)$.

For $p \in (0, 1)$, $\| \ \|_p$ is not subadditive (see p. 4), so if we want a generalization of the above theorem that includes L^p-spaces for $p \in (0, 1)$, then we cannot require subadditivity of $\| \ \|$.
In this section we will show that by dropping the subadditivity condition and allowing $p \in (0, \infty)$ in theorem 1.1, we obtain a generalization of the Kakutani-Bohnenblust characterization:

Theorem 1.2 (Generalization of the K.-B. characterization of L^p)
Let E be a Riesz space, let $p \in (0, \infty)$ and let $|||\ |||$ be separating, absolutely homogeneous, Riesz, and p-additive.
Suppose E is "$|||\ |||$-complete" in the following sense:

$$\left.\begin{array}{l} x_1, x_2, \ldots \in E, \\ |||x_n - x_m||| \to 0 \quad (n, m \to \infty) \end{array}\right\} \implies \exists x \in E \text{ with } |||x - x_n||| \to 0 \quad (n \to \infty).$$

Then there exist a measure space $(S, \mathcal{A}, \mu)$ and a surjective Riesz isomorphism $\Omega : E \to L^p(\mu)$ such that $\|\Omega(x)\|_p = |||x|||$ $(x \in E)$.

Actually, the proof of the above generalization will be obtained as a consequence of another characterization of L^p-spaces:

Theorem 1.3 (σ-Levi characterization of L^p-spaces)
Let E be a Riesz space, let $p \in (0, \infty)$ and let $|||\ ||| : E \to [0, \infty)$ be separating, absolutely homogeneous, Riesz and p-additive.
Suppose E has the σ-Levi property with respect to $|||\ |||$, i.e. if $(u_n)_n$ is a sequence in E^+ with $0 \leq u_n \uparrow$ and $\sup_n |||u_n||| < \infty$ then $\sup_n u_n$ exists in E.
Then there exist a measure space $(S, \mathcal{A}, \mu)$ and a surjective Riesz isomorphism $\Omega : E \to L^p(\mu)$ such that $\|\Omega(x)\|_p = |||x|||$ $(x \in E)$.

The set-up of this section is as follows.
First we will show that if E is weak-Freudenthal (which is the case in both theorem 1.2 and theorem 1.3), then $\rfloor\!\rfloor\!\lfloor\!\lfloor$ satisfies some interesting inequalities: which we will call "$\| \|_p$-inequalities".
Using those inequalities we will explain how the generalized Kakutani-Bohnenblust characterization 1.2 is a consequence of theorem 1.3.
Finally, we will use the $\| \|_p$-inequalities to provide a proof of theorem 1.3.

We begin with taking a closer look at the functional $\rfloor\!\rfloor\!\lfloor\!\lfloor : E \to [0, \infty)$ that appears in the generalization of the Kakutani-Bohnenblust characterization and also in theorem 1.3.
From the fact that it is separating and Riesz we get the following relation between order convergence and $\rfloor\!\rfloor\!\lfloor\!\lfloor$-convergence (familiar from the context of locally solid Riesz spaces, see 0.126 *iii*):

Lemma 1.4
Let E be a Riesz space, and let $\rfloor\!\rfloor\!\lfloor\!\lfloor : E \to [0, \infty)$ be separating and Riesz.
Suppose $x_n \uparrow$ in E, $x \in E$, and $\rfloor\!\rfloor x - x_n \lfloor\!\lfloor \to 0$.
Then $x = \sup_n x_n$.

Proof
(i) x is an upper bound of $\{x_n\}_n$: Let $n \in \mathbb{N}$. For all $m > n$:

$$(x - x_n)^- \le (x - x_m)^- + \underbrace{(x_m - x_n)^-}_{=0} \le |x - x_m| .$$

Hence $\rfloor\!\rfloor (x - x_n)^- \lfloor\!\lfloor \le \liminf_m \rfloor\!\rfloor x - x_m \lfloor\!\lfloor = 0$ i.e. $(x - x_n)^- = 0$.
(ii) x is the least of the upper bounds of $\{x_n\}_n$: Let $w \ge x_n$ (all n). Then for all n:

$$(w - x)^- \le \underbrace{(w - x_m)^-}_{=0} + (x_m - x)^- \le |x_m - x|.$$

Thus, $\rfloor\!\rfloor (w - x)^- \lfloor\!\lfloor \le \liminf_m \rfloor\!\rfloor x - x_m \lfloor\!\lfloor = 0$. □

A tricky point about the functional $\rfloor\!\rfloor\!\lfloor\!\lfloor : E \to [0, \infty)$ is that it is not clear whether it defines a (metrizable) vector space topology, because for what reason should the vector addition be continuous as a map from $E \times E$ to E? To deal with that problem we will impose the weak-Freudenthal property (cf. [La]).

Definition 1.5
A Riesz space E is called weak-Freudenthal if for every $e \in E^+$, $0 \le a \le e$, and $\varepsilon \in (0, \infty)$ there exist disjoint $e_1, \dots, e_n \in E^+$ and scalars $\alpha_1, \dots, \alpha_n$ such that

$$\textstyle\sum_1^n e_n = e\,, \text{ and } |a - \sum_1^n \alpha_n e_n| \le \varepsilon e\,.$$

(Alternatively formulated: if D is a principal ideal of E and $C(\Omega)$ its Yosida-representation, then Ω is zero-dimensional).

The following lemma explains that if E is weak-Freudenthal, then there are 'enough' disjoint elements in E to extend the p-additivity of $\rfloor\!\rfloor\!\lfloor\!\lfloor$ to inequalities familiar from the context of L^p-spaces, which state amongst others that $\rfloor\!\rfloor\!\lfloor\!\lfloor$ is a quasi-norm.

Lemma 1.6 ($\| \|_p$-inequalities)
Let E be a Riesz space, let $p \in (0,\infty)$, and let $⦀⦀ : E \to [0,\infty)$ be absolutely homogeneous, Riesz and p-additive.
Suppose E is weak-Freudenthal.
Then $⦀⦀$ satisfies the following inequalities ("$\| \|_p$-inequalities"):

	for $u,v \in E^+$:	for $x,y \in E$:
$p \in (0,1]$:	$⦀u⦀ + ⦀v⦀ \le ⦀u+v⦀$	$⦀x+y⦀^p \le ⦀x⦀^p + ⦀y⦀^p$
$p \in [1,\infty)$:	$⦀u⦀^p + ⦀v⦀^p \le ⦀u+v⦀^p$	$⦀x+y⦀ \le ⦀x⦀ + ⦀y⦀$

In particular, $⦀⦀$ is a norm for $p \in [1,\infty)$ and a quasi-norm for $p \in (0,1]$ (with quasi-subadditivity factor $2^{(1-p)/p}$).

Proof
First observe that the $\| \|_p$-inequalities (see p. 3) hold for $E = \ell^p$ and $⦀⦀ = \| \|_p$, i.e. for $p \in (0,\infty)$:

$$\begin{aligned} &\|x+y\|_p^{1\wedge p} \le \|x\|_p^{1\wedge p} + \|y\|_p^{1\wedge p} \quad (x,y \in \ell^p)\,; \\ &\|u\|_p^{1\vee p} + \|v\|_p^{1\vee p} \le \|u+v\|_p^{1\vee p} \quad (u,v \in \ell^{p+})\,. \end{aligned} \tag{$\sharp$}$$

Let now $a,b \in E^+$, and let $\varepsilon \in (0,\infty)$. Since E is weak-Freudenthal there exist (see 0.66) $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \in (0,\infty)$ and $e_1,\dots,e_n \in E^+$ disjoint such that

$$\begin{aligned} &\textstyle\sum_{\iota=1}^n e_\iota = a+b\,; \\ &\textstyle\sum_{\iota=1}^n (\alpha_\iota - \varepsilon)^+ e_\iota \le a \le \sum_{\iota=1}^n \alpha_\iota e_\iota\,; \\ &\textstyle\sum_{\iota=1}^n (\beta_\iota - \varepsilon)^+ e_\iota \le b \le \sum_{\iota=1}^n \beta_\iota e_\iota\,. \end{aligned}$$

Using that $⦀⦀$ is p-additive, absolutely homogeneous, and Riesz, the above inequalities imply that $\left\|(⦀e_\iota⦀)_{\iota=1}^n\right\|_p = ⦀a+b⦀$ and

$$\begin{aligned} \left\| ([\alpha_\iota - \varepsilon]^+ \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p &\le ⦀a⦀ \le \left\| (\alpha_\iota \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p \\ \left\| ([\beta_\iota - \varepsilon]^+ \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p &\le ⦀b⦀ \le \left\| (\beta_\iota \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p \\ \left\| ([\alpha_\iota + \beta_\iota - 2\varepsilon]^+ \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p &\le ⦀a+b⦀ \le \left\| ([\alpha_\iota + \beta_\iota] \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p\,. \end{aligned} \tag{$\flat$}$$

Using the above estimates for $⦀a⦀$, $⦀b⦀$, $⦀a+b⦀$, and ($\sharp$) we infer

$$\begin{aligned} &⦀a⦀^{1\vee p} + ⦀b⦀^{1\vee p} \\ &\underset{(\flat)}{\le} \left\| (\alpha_\iota \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p^{1\vee p} + \left\| (\beta_\iota \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p^{1\vee p} \\ &\underset{(\sharp)}{\le} \left\| ([\alpha_\iota + \beta_\iota] \cdot ⦀e_\iota⦀)_{\iota=1}^n \right\|_p^{1\vee p} \\ &\le \left\| ([\alpha_\iota + \beta_\iota - 2\varepsilon]^+ ⦀e_\iota⦀)_{\iota=1}^n + (2\varepsilon\, ⦀e_\iota⦀)_{\iota=1}^n \right\|_p^{1\vee p} \\ &\underset{(\sharp)}{\le} \left[\left\|([\alpha_\iota + \beta_\iota - 2\varepsilon]^+ ⦀e_\iota⦀)_{\iota=1}^n\right\|_p^{1\wedge p} + \left\|(2\varepsilon\, ⦀e_\iota⦀)_{\iota=1}^n\right\|_p^{1\wedge p} \right]^{(1\vee p)/(1\wedge p)} \\ &\underset{(\flat)}{\le} \left[⦀a+b⦀^{1\wedge p} + (2\varepsilon)^{1\wedge p}\, ⦀a+b⦀^{1\wedge p} \right]^{(1\vee p)/(1\wedge p)} \end{aligned}$$

Since $\varepsilon \in (0,\infty)$ is arbitrary: $⦀a⦀^{1\vee p} + ⦀b⦀^{1\vee p} \le ⦀a+b⦀^{1\vee p}$ $(a,b \in E^+)$.

A similar argument shows $|||a+b|||^{1\wedge p} \le |||a|||^{1\wedge p} + |||b|||^{1\wedge p}$ $(a,b \in E^+)$. Since $|||\ |||$ is Riesz, the latter inequality extends to $a, b \in E$. □

The following lemma clarifies why theorem 1.2 is a consequence of theorem 1.3.

Lemma 1.7

Let E be a Riesz space, let $p \in (0,\infty)$ and let $|||\ |||$ be separating, Riesz and p-additive. Then we have the following implications:

E is $|||\ |||$-complete $\implies$ E is σ-Levi $\implies$ E is super-Dedekind complete.

(Here $|||\ |||$-completeness is taken in the same sense as in theorem 1.2).

Proof

(I) E is $|||\ |||$-complete $\Rightarrow$ E is σ-Levi

Let E be $|||\ |||$-complete. Then E is conditionally σ-laterally complete.

◁Indeed, let $(u_n)_n$ be a disjoint sequence in E^+ bounded above by $u \in E^+$. By p-additivity, $\sum_1^\infty |||u_i|||^p \le |||u|||^p < \infty$. Consequently, the sums $s_n := u_1 + \ldots + u_n$ form a $|||\ |||$-Cauchy sequence: $|||s_m - s_n|||^p \le \sum_{i>N} |||u_i|||^p \to 0$ (if $m \ge n \ge N \to \infty$). Hence, there exists an $s \in E$ such that $\lim_n |||s - s_n||| = 0$. Since $s_n \uparrow$, lemma 1.4 implies that $s = \sup_n s_n$.▷

Since E is conditionally σ-laterally complete, $|||\ |||$ satisfies $\|\ \|_p$-inequalities (use lemma 1.6 and 0.64), which allow us to see that E is σ-Levi: if $0 \le u_n \uparrow$ in E and $\sup_n |||u_n||| < \infty$, then for $m \ge n \ge N$:

$$|||u_m - u_n|||^{1\vee p} \le |||u_m|||^{1\vee p} - |||u_n|||^{1\vee p} \le \sup_k |||u_k|||^{1\vee p} - \sup_{1\le k\le n} |||u_k|||^{1\vee p},$$

so $(u_n)_n$ is $|||\ |||$-Cauchy, and its $|||\ |||$-limit is its supremum by lemma 1.4 again.

(II) E is σ-Levi $\implies$ E is super-Dedekind complete

Let E be σ-Levi. Then E is σ-Dedekind complete (0.14), and $|||\ |||$ satisfies $\|\ \|_p$-inequalities (0.64 and 1.6).

To show that E is super-Dedekind complete, let $U_0 \subset E^+$ be bounded from above by $v \in E^+$. Set $U := \{\sup_k u_k : u_1, u_2, \ldots \in U_0\}$ and observe that U has the same upper bounds as U_0 and that $\sup_k u_k \in U$ whenever $u_k \in U$ (all k).

Realizing that $\sigma := \sup\{|||u||| : u \in U\} \le |||v||| < \infty$, we choose inductively $u_1, u_2, \ldots$ in U such that $0 \le u_n \uparrow$ and $|||u_n||| \uparrow \sigma$. Then $u_\infty := \sup_n u_n \in U$, and $|||u_\infty||| = \sigma$: indeed, $|||u_\infty||| \le \sigma$ because $u_\infty \in U$.

We show that $u_\infty = \max U = \sup U = \sup U_0$.

To this end, let $u \in U$. Using a $\|\ \|_p$-inequality, and using that $u \vee u_\infty \in U$ we see that

$$|||u \vee u_\infty - u_\infty|||^{1\vee p} \le |||u \vee u_\infty|||^{1\vee p} - |||u_\infty|||^{1\vee p} \le \sigma^{1\vee p} - |||u_\infty|||^{1\vee p} = 0,$$

so that $u_\infty = u \vee u_\infty \ge u$. □

We now turn to the proof of theorem 1.3.

The idea of the proof is illustrated by way of the following diagram.

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d, "\Phi_p"'] & L^p(\mu) \\ E_1 \arrow[r, "\Psi"'] & L^1(\mu) \arrow[u, "\Phi_{1/p}"'] \end{tikzcd}$$

By taking "p-th powers" of elements of E, we form an abstract L-space E_1, which is isometrically Riesz isomorphic to an $L^1(\mu)$ (the K.-B. theorem for $p = 1$). By taking "p-th roots" of elements of $L^1(\mu)$, we obtain $L^p(\mu)$. The resulting map is an isometric Riesz isomorphism from E onto $L^p(\mu)$.

Proof of theorem 1.3

(I) E is an ideal of $C^\infty(S)$

Using the Maeda-Ogasawara-Vulikh representation theorem (0.30), we may view E as an order dense Riesz subspace of an $C^\infty(S)$. Since E is Dedekind complete, E is an ideal of $C^\infty(S)$ (see 0.80).

(II) The definition of E_1

The map $\overline{\phi}_p : [-\infty, \infty] \to [-\infty, \infty]$ defined by

$$\overline{\phi}_p(t) := \begin{cases} t^p & \text{if } t \in [0, \infty) ; \\ -|t|^p & \text{if } t \in (-\infty, 0) ; \\ +\infty & \text{if } t = \infty ; \\ -\infty & \text{if } t = -\infty , \end{cases}$$

is an order preserving homeomorphism $[-\infty, \infty] \to [-\infty, \infty]$.

As a result

$$\begin{aligned} f \in C^\infty(S) \quad &\Rightarrow \quad \Phi_p(f) = f^p := \overline{\phi}_p \circ f \ \in C^\infty(S) \\ f \in C^\infty(S) \quad &\Rightarrow \quad |f^p| \le |f|^p . \end{aligned}$$

Set

$$E_1 := \{f^p : f \in E\} , \qquad \rVert f^p \Vert_1 := \rVert f \Vert^p \quad (f \in E) .$$

(III) E_1 is a Riesz ideal of $C^\infty(S)$ that is σ-Levi with respect to $\rVert\,\Vert_1$, and $\rVert\,\Vert_1$ is 1-additive

That E_1 is closed under scalar multiplication is straightforward from its definition. Further, E_1 inherits the solidness of E, because Φ_p is an order automorphism of $C^\infty(S)$, so that e.g. $|\Phi(f)| \le \Phi(|f|)$ $(f \in C^\infty(S))$ for $\Phi = \Phi_p$ and $\Phi = \Phi_p^{-1}$. For the same reason (and the very definition of $\rVert\,\Vert_1$), the σ-Levi property of $(E, \rVert\,\Vert)$ passes on to $(E_1, \rVert\,\Vert_1)$, while the p-additivity of $\rVert\,\Vert$ translates in the 1-additivity of $\rVert\,\Vert_1$. Finally, E_1 is closed under addition: if $f, g \in E$, then

$$|f^p + g^p| \le |f|^p + |g|^p \le 2(\underbrace{|f| + |g|}_{\in E})^p , \text{ so } f^p + g^p \in E_1 , \text{ since } E_1 \text{ is solid.}$$

(IV) $(E_1, \rVert\,\Vert_1) \simeq (L^1(\mu), \|\ \|_1)$

By the K.-B.characterization (0.111) for $p = 1$, there exists a measure space $(S, \mathcal{A}, \mu)$ and a Riesz isomorphism $\Psi : E_1 \to L^1(\mu)$ such that $\|\Psi(h)\|_1 = \rVert h \Vert_1$ $(h \in E_1)$.

(V) $L^p(\mu)$ is order isomorphic to $L^1(\mu)$

Define $\Phi_{1/p} : L^1(\mu) \to L^p(\mu)$ by $\Phi_{1/p}(f) = f^{1/p} := \phi_{1/p} \circ f$ $(f \in L^1(\mu))$, where $\phi_{1/p}$ is the restriction of $\overline{\phi}_{1/p}$ to $\mathbb{R}$, which is an order preserving homeomorphism $\mathbb{R} \to \mathbb{R}$. Then $\Phi_{1/p}$ is an order isomorphism and $\|f^{1/p}\|_p^p = \|f\|_1$ $(f \in L^1(\mu))$.

(VI) $\Omega := \Phi_{1/p}\Psi\Phi_p$ is an isometric Riesz isomorphism $(E, \rVert\,\Vert) \to (L^p(\mu), \|\ \|_p)$.

The relations between $\rVert\,\Vert$, $\rVert\,\Vert_1$, $\|\ \|_1$ and $\|\ \|_p$ above yield that $\|\Omega(x)\|_p = \rVert x \Vert$ $(x \in E)$. Further, E is uniformly complete (0.56), and Ω satisfies:

$$\begin{aligned} \Omega(ru) &= r\Omega(x) && (r \in [0, \infty) , u \in E^+) ; \\ \Omega(u \vee v) &= \Omega(u) \vee \Omega(v) && (u, v \in E^+) ; \\ \Omega(u \wedge v) &= \Omega(u) \wedge \Omega(v) && (u, v \in E^+) , \end{aligned}$$

which implies that Ω is a Riesz isomorphism in view of lemma 1.8 below. □

Lemma 1.8
Let E, and F be uniform complete, Archimedean Riesz spaces
Suppose that $\Omega : E^+ \to F^+$ is a map satisfying:

(i) $\Omega(ru) = r\Omega(u)$ $(r \in [0, \infty)\,,\ u \in E^+)$;

(ii) $\Omega(u \vee v) = \Omega(u) \vee \Omega(v)$ $(u, v \in E^+)$;

(iii) $\Omega(u \wedge v) = \Omega(u) \wedge \Omega(v)$ $(u, v \in E^+)$.

Then

i. *Ω is additive, and*
ii. *Ω can be uniquely extended to a Riesz homomorphism from E into F.*

Proof
We prove that Ω is additive. The rest of the claim is a consequence thereof.
Let $a, b \in E^+$. Since Ω preserves the ordering, Ω maps $E^+_{[a+b]}$ into $F^+_{[\Omega(a+b)]}$. Let $C(X)$ be a Yoshida representation space of $E_{[a+b]}$ and $C(Y)$ one of $F_{[\Omega(a+b)]}$, and view Ω as a map from $C(X)^+$ to $C(Y)^+$ (0.54).
For every Riesz homomorphism $\psi : C(Y) \to \mathbb{R}$, the map $\psi \circ \Omega : C(X)^+ \to \mathbb{R}$ is positively homogeneous, $\vee$-preserving, and $\wedge$-preserving. By lemma 0.7 it is therefore additive, whence $\psi(\Omega(a + b)) = \psi(\Omega(a)) + \psi(\Omega(b))$. Since the Riesz homomorphisms of $C(Y)$ are separating (0.7), $\Omega(a + b) = \Omega(a) + \Omega(b)$. □

Remarks 1.9
Ad theorem 1.3 For the proof of 1.3 to work, it is not necessary to know in advance that E is Dedekind complete (and hence an ideal of $C^\infty(S)$): it suffices that E is a uniformly complete Riesz space. Indeed, using the functional calculus for uniformly complete Riesz spaces ([BdPvR]), one can deduce that for $f, g \in E$ there is a sequence in the Riesz space generated by f and g that converges $(|f| + |g|)$-uniformly to $(f^p + g^p)^{1/p}$.

Ad lemma 1.7 If $[\![\,]\!]$ satisfies $\| \|_p$-inequalities, then the σ-Levi property and $[\![\,]\!]$-completeness are equivalent (see 0.105).

Ad definition 1.5 In [La, p. 411-417], B. Lavrič presents some characterizations of Archimedean Riesz spaces with the weak-Freudenthal property, which he calls Riesz spaces in which the weak form of Freudenthal's spectral theorem holds (see 0.63).

Weak-Freudenthal property The weak-Freudenthal property is weaker than the principal projection property, but stronger than the property of having sufficiently many projections.

1.2 $\| \|_p$-dense Riesz subspaces of L^p

As an intermezzo (and a tool for the next section) we present some sufficient conditions on a Riesz space to be isometrically Riesz isomorphic to a $\| \|_p$-dense Riesz subspace of an L^p-space.

Theorem 1.10
Let E be a Riesz space, let $p \in (0, \infty)$ and let $\|\!\|\ : E \to [0, \infty)$ be separating, absolutely homogeneous, Riesz and p-additive.
Suppose one of the following conditions holds,

(i) E is weak-Freudenthal,

(ii) $\|\!\|$ is σ-Fatou.

Then there exist a measure space $(S, \mathcal{A}, \mu)$ and a Riesz isomorphism $\Omega : E \to L^p(\mu)$ such that $\|\Omega(x)\|_p = \|x\|$ $(x \in E)$ and $\Omega(E)$ is a $\|\ \|_p$-dense Riesz subspace of $L^p(\mu)$.

The idea is that either of the above conditions forces L^p to be the completion of E as quasi-normed Riesz space. The latter is a straightforward generalization of the notion of a normed Riesz space, and will be discussed first.

Definition 1.11
A quasi-normed Riesz space is a pair $(E, \|\!\|)$ where E is a Riesz space and $\|\!\|$ is a Riesz quasi-norm, i.e. $\|\!\|\ : E \to [0, \infty)$ is separating, absolutely homogeneous, Riesz and

there exists a constant $K \in [1, \infty)$ such that
$\|x + y\| \le K[\ \|x\| + \|y\|\ ]$ for all $x, y \in E$. (quasi-subadditivity)

By the quasi-subadditivity, a Riesz quasi-norm induces a locally solid, locally bounded vector space topology.
A quasi-Banach lattice is a quasi-normed Riesz space that is complete with respect to its Riesz quasi-norm.

Every quasi-normed Riesz space is homeomorphic (and Riesz isomorphic) to a dense Riesz subspace of a quasi-Banach lattice:

Lemma 1.12
Let $(E, \|\!\|)$ be a quasi-normed Riesz space.
Then there exist a $K \in [1, \infty)$, a quasi-Banach lattice $(\hat{E}, \|\!\|^\wedge)$, and a Riesz isomorphism $\iota : E \to \hat{E}$ such that $\iota(E)$ is $\|\!\|^\wedge$-dense in $\hat{E}$ and

$$K^{-1}\|\iota(x)\|^\wedge \le \|x\| \le K\|\iota(x)\|^\wedge \quad (x \in E).$$

If $\|\!\|$ is r-subadditive for some $r \in (0, 1]$, then we can take $K = 1$ i.e. in this case $\|\ \|^\wedge$ is an extension of $\|\!\|$ to $\hat{E}$.

The quasi-Banach lattice $(\hat{E}, \|\!\|^\wedge)$ is called a completion of the quasi-normed Riesz space $(E, \|\!\|)$.

◁ For the existence of such completion, observe that for an equivalent r-subadditive Riesz quasi-norm $\|\!\|_{(r)}$ (see 0.122) the formula $d(x, y) = \|x - y\|_{(r)}^r$ $(x, y \in E)$ defines a metric on E. By continuity we can extend the vector and lattice operations to the metric space completion $(\hat{E}, \hat{d})$. With Riesz quasi-norm $\|\hat{x}\|^\wedge := \hat{d}(x, 0)$, the pair $(\hat{E}, \|\!\|^\wedge)$ forms a quasi-Banach lattice satisfying the requirements. ▷

We are now prepared for the proof of theorem 1.10.

Proof of theorem 1.10
We will show that under either condition, $\|\!\|$ satisfies $\|\ \|_p$-inequalities. As a result

of the latter, $(E,]\!]\,[\![)$ is a quasi-normed Riesz space with $(1 \wedge p)$-subadditive quasi-norm, its completion $(\hat{E},]\!]\,[\![\hat{\ })$ is a quasi-Banach lattice, and since the p-additivity of $]\!]\,[\![$ passes on to $]\!]\,[\![\hat{\ }$, the latter is isometrically Riesz isomorphic to an L^p-space (theorem 1.2 p. 46).

Ad (i): The weak-Freudenthal property implies $\| \|_p$-properties (see lemma 1.6).

Ad (ii): First observe that the p-additivity of $]\!]\,[\![$ implies that every majorized disjoint system in E^+ is countable. By a lemma of Fremlin (0.58), E therefore has the countable sup property: if a set has a supremum in E, then it contains a *countable* subset having that supremum. Together with the σ-Fatou property, the countable sup property implies that $]\!]\,[\![$ has in fact the Fatou property. The latter ensures that for $x^\delta \in E^\delta$ (the Dedekind completion, see 0.67 on p. 22)

$$]\!]x^\delta[\![^\delta := \sup_\alpha]\!]u_\alpha[\![\quad \text{where } (u_\alpha)_\alpha \text{ is a(ny) net in } E^+ \text{ with } u_\alpha \uparrow |x^\delta|,$$

is well-defined. By its very definition, it is clear that $]\!]\,[\![^\delta$ extends $]\!]\,[\![$, and that $]\!]\,[\![^\delta$ is absolutely homogeneous, Riesz and p-additive. Applying *(i)*, $]\!]\,[\![^\delta$ and hence $]\!]\,[\![$ satisfies $\| \|_p$ inequalities. □

Remarks 1.13

Ad theorem 1.10

- Neither the weak-Freudenthal property of E, nor the σ-Fatou property of $]\!]\,[\![$ are necessary conditions for $(E,]\!]\,[\![)$ to be isometrically Riesz isomorphic to a $\| \|_p$-dense subspace of an L^p:

  (a) $C[0,1]$ is a $\| \|_1$-dense Riesz subspace of $L^1[0,1]$ that is not weak-Freudenthal.

  (b) Take the measure space $S = [0,1]$, $\mathcal{A}$ the Borel σ-algebra of $[0,1]$, and μ the sum of the Lebesgue measure λ and the Dirac measure δ at 0. Then $L^1(\mu)$ consists of the Lebesgue-measurable functions on $[0,1]$ with identification of functions that are equal λ-almost everywhere *and* take equal values at 0. For $f \in L^1(\mu)$, $\int f d\mu = f(0) + \int f d\lambda$. The Riesz subspace $C[0,1]$ is $\| \|_1$-dense in $L^1(\mu)$, but $\| \|_1 |_{C[0,1]}$ is not σ-Fatou.

  ◁The increasing sequence $f_n(t) = nt \wedge 1$ $(t \in [0,1])$, $n \in \mathbb{N}$, has $\mathbb{1}$ as its supremum, but $\|f_n\|_1 = 1 - (2n)^{-1} < 2 = \|\mathbb{1}\|_1$. Further, if $f \in L^1(\mu)$ then there exist $f_n \in C[0,1]$ such that $\int |f - f_n| d\lambda \to 0$ if $n \to \infty$. By modifying f_n at a small interval around 0, we can arrange that $f_n(0) = f(0)$ Then $\|f - f_n\|_1 \to 0$ if $n \to \infty$ ▷

- A sufficient and necessary condition on $(E,]\!]\,[\![_0)$ to be isometrically Riesz isomorphic to a $\| \|_p$-dense Riesz subspace of an L^p is that $(E,]\!]\,[\![_0)$ is a quasi-normed Riesz space such that $]\!]\,[\![_0$ is p-additive and uniformly continuous on $]\!]\,[\![_0$-bounded subsets of E.

  ◁We only prove sufficiency choose an r-subadditive Riesz quasi-norm $]\!]\,[\![$ on E that is equivalent to $]\!]\,[\![_0$ Then $]\!]\,[\![_0$ is uniformly continuous on $]\!]\,[\![$-bounded subsets of E Let $(\hat{E},]\!]\,[\![\hat{\ })$ be the completion of $(E,]\!]\,[\![)$ as quasi-normed Riesz space. For $\xi \in \hat{E}$, the expression

$$]\!]\xi[\![_0\hat{\ } = \lim_n]\!]x_n[\![_0 \quad \text{where } x_n \in E \text{ and }]\!]\xi - x_n[\![\hat{\ } \to 0 \text{ if } n \to \infty,$$

  is well-defined, because sequences in E that $]\!]\,[\![\hat{\ }$-converge to ξ are $]\!]\,[\![$-bounded in E, and $]\!]\,[\![_0$ is uniformly continuous on $]\!]\,[\![$-bounded sets. As a heritage of $]\!]\,[\![_0$, $]\!]\,[\![_0\hat{\ }$ is a p-additive Riesz quasi-norm on $\hat{E}$ equivalent to $]\!]\,[\![\hat{\ }$. Hence, $(\hat{E},]\!]\,[\![_0\hat{\ })$ is isometrically Riesz isomorphic to an L^p-space by theorem 1.2 ▷

- Let λ be the Lebesgue measure on $[0,1]$. The example of $E = C[0,1]$, and

$$\|f\| := \begin{cases} \int_{[0,1]} |f| d\lambda & \text{if } f \text{ has a zero,} \\ 2\int_{[0,1]} |f| d\lambda & \text{if } f \text{ has no zeroes,} \end{cases}$$

shows that requiring $\|\ \|$ to be separating, absolutely homogeneous, Riesz and 1-additive is not sufficient to conclude that there is a measure space $(S, \mathcal{A}, \mu)$ such that E is $\|\ \|_1^{(\mu)}$-dense in $L^1(\mu)$ *and* $\|\ \| = \|\ \|_1^{(\mu)}\,|_E$: $\|\ \|$ is not additive on E^+.

Order denseness versus $\|\ \|_p$-denseness For a Riesz subspace of L^p, order denseness implies $\|\ \|_p$-denseness (by the Lebesgue property of $\|\ \|_p$, see 0.137), but not vice versa: $C[0,1]$ is a $\|\ \|_1$-dense Riesz subspace of $L^1[0,1]$ that is not order dense in $L^1[0,1]$.

1.3 Another Riesz isometric characterization of L^p

In theorem 1.3, we used the σ-Levi property to characterize L^p. In this section, we try to weaken that condition: if E is σ-Levi, then E is uniformly complete (by σ-Dedekind completeness) and the σ-Levi property holds for increasing sequences induced by disjoint positive sequences. The following theorem tells us that these properties alone are enough to characterize L^p-spaces.

Theorem 1.14
Let E be a Riesz space and let $\|\ \| : E \to [0,\infty)$ be separating, absolutely homogeneous, Riesz, and p-additive for some $p \in (0,\infty)$.
Suppose that

(i) E is disjoint σ-Levi with respect to $\|\ \|$, i.e.

$$\left.\begin{array}{l} e_1, e_2, \ldots \in E^+ \text{ disjoint}, \\ \sum_1^\infty \|\sum_1^n e_i\|^p < \infty \end{array}\right] \quad \Rightarrow \quad \sup_n e_n \in E\,;$$

(ii) E is uniformly complete.

Then there are a measure space $(S, \mathcal{A}, \mu)$ and a surjective Riesz isomorphism $\Omega : E \to L^p(\mu)$ such that $\|\Omega(x)\|_p = \|x\|$ (all $x \in E$).

The ideas behind the proof of theorem 1.14 can be put into a more general context yielding:

Theorem 1.15
Let (E, τ) be a Hausdorff locally solid Riesz space.
Suppose that

$$\left.\begin{array}{l} (u_n)_1^\infty \text{ disjoint in } E^+, \\ \sum_n u_n\ \tau\text{-bounded} \end{array}\right] \Rightarrow \sum_n u_n \xrightarrow{\tau} \sup_n u_n\,. \qquad \text{(strongly disjoint } \sigma\text{-Levi)}$$

Then

i. the topological completion $(\hat{E}, \hat{\tau})$ is a Lebesgue Riesz space (therefore Dedekind complete);

ii. *if $\{u_n\}_1^\infty$ is a disjoint countable subset of E^+ that is order bounded in $\hat{E}^+$, then $\hat{E}$-$\sup_n u_n \in E$;*

as a consequence (0.34 and 0.64): for $u, v \in E$ the principal projection $P_u(v)$ ($\hat{P}_u(v)$) of v onto the band generated by u in E (in $\hat{E}$ respectively) exists and

$$P_u(v) = \hat{P}_u(v) \in E^+ \quad (u, v \in E^+) .$$

Assume moreover that τ is metrizable (equivalently, there is a countable 0-neighborhood basis).
Then:

iii. *$\hat{E}$ is super-Dedekind complete;*

iv. *components of elements of E^+ in $\hat{E}$ are in E;*

v. *E is majorizing and order dense in $\hat{E}$.*

Conclusion (under the above conditions):

vi. *if $\hat{e} \in \hat{E}$, then there is a sequence in E that converges relatively uniformly to $\hat{e}$ with respect to a regulator in E (in this sense $\hat{E}$ is a relatively uniform completion of E);*

vii. *E is topologically complete if (and only if) E is uniformly complete;*

viii. *$\hat{E}$ is the (σ-)Dedekind completion of E.*

Let us first show that theorem 1.15 is indeed a generalization of theorem 1.14.

Proof of theorem 1.14 from theorem 1.15

Since E is disjoint σ-Levi with respect to $|||\,|||$ and $|||\,|||$ is p-additive, E is conditionally σ-laterally complete. Thus lemma 1.10 (p. 52) allows us to identify E with a $\|\,\|_p$-dense Riesz subspace of $L^p(\mu)$ for a suitable measure space $(S, \mathcal{A}, \mu)$ and $|||\,|||$ with $\|\,\|_p\,|_E$. In particular, E is a *metrizable* Hausdorff locally solid Riesz space. To see that also the first condition of theorem 1.15 is satisfied (i.e. E is strongly disjoint σ-Levi), we show

$$\left.\begin{array}{l}(e_n)_n \text{ disjoint in } E^+,\\ \sum_n e_n \;\; \|\,\|_p\text{-bounded}\end{array}\right\} \Rightarrow \sum_n e_n \to \sup_n e_n \begin{pmatrix}\text{relatively uniformly with}\\ \text{respect to a regulator in } E\end{pmatrix}.$$

$\triangleleft$Indeed, let $(e_n)_n$ be as in the premiss. By the σ-Levi property of L^p: $\hat{e} := L^p$-$\sup_n e_n$ exists; therefore $\sum_1^\infty \| e_n \|_p^p \le \| \hat{e} \|_p^p < \infty$, and so $e := E$-$\sup_n e_n$ exists by the disjoint σ-Levi property of E (assumption i. of theorem 1.14).
Choose $r_n \in [1, \infty)$ such that $1 \le r_n \uparrow \infty$ and $\sum_1^\infty \| r_n e_n \|_p^p < \infty$. By the disjoint σ-Levi property of E (i. of theorem 1.14) again, E-$\sup_n r_n e_n$ exists in E, and

$$0 \le e - \sum_1^N e_n = E\text{-}\sup_{n>N} e_n \le \frac{1}{r_{N+1}} (E\text{-}\sup_{n>N} r_n e_n) \le \frac{1}{r_{N+1}} (E\text{-}\sup_n r_n e_n) .$$

$\triangleright$

Using theorem 1.15, the uniform completeness of E implies that $E = L^p(\mu)$. □

We now discuss the proof of theorem 1.15. The crux of the argument is that components of elements of E^+ in $\hat{E}$ are already contained in E. From there, we can proceed using the order denseness lemma and the dominance lemma below.

Lemma 1.16 (Order denseness)
Let E be a Riesz subspace of a Riesz space F.
Suppose

(i) *the disjoint complement of E in F is null, i.e. $\{x \in F : x \perp E\} = \{0\}$;*

(ii) *F has the principal projection property;*

(iii) *all components of elements of E^+ in F are already in E.*

Then E lies order densely in F.

Lemma 1.17 (Dominance)
Let E be a Riesz subspace of a Riesz space F.
Suppose

(i) *E is order dense in F;*

(ii) *if D is a disjoint system in E^+ that is majorized in F, then F-$\sup D \in E$.*

Then E majorizes F.

We postpone the proofs of lemmata 1.16 and 1.17 to the end of this section (page 58), and first show how theorem 1.15 is proved using them.

Proof of theorem 1.15
(0) Some preparatory observations
We first observe that the strong disjoint σ-Levi property of (E, τ) implies

$$\left.\begin{array}{l} u_1, u_2, \ldots \in E^+ \text{ disjoint}, \\ \hat{v} \in \hat{E}^+, u_n \le \hat{v} \text{ all } n \in \mathbb{N} \end{array}\right] \Rightarrow \textstyle\sum_n u_n \xrightarrow{\hat{\tau}} \hat{E}\text{-}\sup_n u_n = E\text{-}\sup_n u_n . \qquad (*)$$

◁Let $(u_n)_n$ and $\hat{v}$ as in the premiss. Then $(u_n)_n$ is order bounded in $\hat{E}$, whence $\hat{\tau}$-bounded so τ-bounded (0.126 ii). By assumption, $\sum_n u_n$ then τ-converges to E-$\sup_n u_n$ But then $\sum_n u_n$ is also an upwards directed, $\hat{\tau}$-convergent sequence in $\hat{E}$, and can as such only have $\hat{E}$-$\sup_n u_n$ as its limit (0.126 iii) Thus $\hat{E}$-$\sup_n u_n = E$-$\sup_n u_n$ ▷

A first consequence of $(*)$ is that majorized disjoint sequences in E^+ τ-converge to 0, so that (E, τ) is pre-Lebesgue by criterion 0.136. Then $(\hat{E}, \hat{\tau})$ is a Lebesgue Riesz space, so $\hat{E}$ is Dedekind complete (0.140) and $\hat{\tau}$ is Fatou (0.138).
As a second consequence of $(*)$, $P_u(v) = \hat{P}_u(v)$ for $u, v \in E^+$.

◁Indeed, using lemma 0.34 there exist $u_n \in E (\subset \hat{E})$ such that $u_n \wedge u_m = 0$ if $|n - m| \ge 2$ and $P_u(v) = E$-$\sup_n u_n = \hat{E}$-$\sup_n u_n = \hat{P}_u(v)$ (cf. the proof of lemma 0 64).▷

Finally, because of the Lebesgue property of $\hat{E}$ (see 0.140)

if τ metrizable, then $\hat{E}$ is a super-Dedekind complete Riesz space. (%)

Assuming for the rest of the proof that τ is metrizable, we proceed.
(I) Components of elements of E^+ in $\hat{E}$ are in E
Let $e \in E^+$, and let for $F = E$ and $F = \hat{E}$, respectively,

$$C_F(e) := \{u \in F \ : \ u \wedge (e - u) = 0\} ,$$

be the lattice of components of e in F. Observe that $C_{\hat{E}}(e)$ inherits Dedekind completeness of $\hat{E}$.

We first show that

$$\{u_n\}_{n\in\mathbb{N}} \subset C_E(e) \quad \Longrightarrow \quad \hat{E}\text{-}\sup_n u_n\,,\ \hat{E}\text{-}\inf_n u_n \in C_E(e)\,. \tag{b}$$

i.e. $C_E(e)$ is a σ-Dedekind complete, regular (sub)lattice of $\hat{E}$.

Proof. Let $\{u_n\}_n \subset C_E(e)$ Define

$$e_n := \sup_{k\le n} u_k - \sup_{k<n} u_k \quad (n \ge 2) \quad \text{and } e_1 = u_1$$

Then $\{e_n\}_n$ is a disjoint system in $C_{\hat{E}}(e)$.

◁Use that a supremum of a system of components of e is a component of e, and that $w - v$ is a component of e, disjoint from v, if $w \ge v$ are components of e (see 0.41) ▷

Further, we have for all $n \in \mathbb{N}$ that

$$P(n)\colon \begin{cases} e_n \in E \quad (\text{i.e. } e_n \in C_E(e)\ )\,; \\ \hat{E}\text{-}\sup_{k\le n} e_k = \hat{E}\ \sup_{k\le n} u_k\,, \end{cases}$$

holds.

Conclusion: $\hat{E}\text{-}\sup_n u_n = \hat{E}\text{-}\sup_n e_n \in E$ by the disjoint σ-Levi property (see $(*)$).

The assertion for infima follows from the equality (see 0.41)

$$\hat{E}\text{-}\inf_n u_n = e - \hat{E}\text{-}\sup_n (e - u_n)$$

In view of the super-Dedekind completeness of $\hat{E}$, the σ-Dedekind completeness of $C_E(e)$ yields in fact the Dedekind completeness of $C_E(e)$:

$$\{u_\alpha\}_\alpha \subset C_E(e) \quad \Longrightarrow \quad \hat{E}\text{-}\sup_\alpha u_\alpha\,,\ \hat{E}\text{-}\inf_\alpha u_\alpha \in C_E(e)\,. \tag{\&}$$

Using (&) and the Fatou property of $\hat{\tau}$, we now show that $C_{\hat{E}}(e) = C_E(e)$.

Let $\hat{u} \in C_{\hat{E}}(e)$. We choose a net $(u_\alpha)_\alpha$ in E^+ such that $u_\alpha \xrightarrow{\hat{\tau}} \hat{u}$. Actually, by replacing u_α by $P_{(u_\alpha - e/2)^+}(e)$ we can arrange that $u_\alpha \in C_E(e)$ (all α).

◁Fix α and set $u := u_\alpha$. Then one elementarily verifies in a Yoshida-representation (of the ideal generated by e in $\hat{E}$) with $e \mapsto \mathbb{1}$ that the function

$$|P_{(u_\alpha - e/2)^+}(e) - \hat{u}| = |\hat{P}_{(u-e/2)^+}(e) - \hat{P}_{\hat{u}}(e)| = |\mathbb{1}_{\overline{[u>1/2]}} - \mathbb{1}_{\overline{[\hat{u}>0]}}|$$

is $\{0,1\}$-valued and $\le 2|u - \hat{u}| = 2|u_\alpha - \hat{u}|$ ($\hat{u}$ is $\{0,1\}$-valued).▷

By (&) and lemma 0.133, there is a $u \in C_E(e)$ such that $u_\alpha \xrightarrow{\tau} u$.

Then $\hat{u} = \hat{\tau}\text{-}\lim_\alpha u_\alpha = u \in C_E(e)$.

(II) E is order dense in $F := \hat{E}$.

This is an application of lemma 1.16 with $F := \hat{E}$.

◁Indeed, the last condition of lemma 1.16 is satisfied in view of (I) above Further, the principal projection property follows from the Dedekind completeness of $\hat{E}$. Finally, $E^d = \{0\}$, because E is topologically dense in $\hat{E}$, and the lattice operations are continuous. $E \perp E^d$ gives $\hat{E} = \overline{E} \perp E^d$, so $E^d \subset \hat{E}^d = \{0\}$.▷

(III) E is majorizing in $\hat{E}$

In view of (II), this follows from lemma 1.17 after we have shown that

if D is a disjoint system in E^+ that is majorized in $\hat{E}$,
then $\hat{E}$-sup $D \in E$. (♯)

To this end, let $D \subset E^+$ be a disjoint system that has a majorant in $\hat{E}$. By the super-Dedekind completeness of $\hat{E}$, D is countable and thus $\hat{E}$-sup $D \in E$ by $(*)$.

(IV) For every element $\hat{e} \in \hat{E}$ there is a sequence in E that converges relatively uniformly to $\hat{e}$ with respect to a regulator in E.

Let $\hat{e} \in \hat{E}^+$. By (III) there is an $e \in E^+$ such that $\hat{e} \leq e$. Because $\hat{E}$ is weak-Freudenthal, there exists by lemma 0.66 a sequence of linear combinations of elements of $C_{\hat{E}}(e) = C_E(e)$ that converges e-uniformly to $\hat{e}$.

The rest of the conclusion of theorem 1.15 is elementarily proven. □

We complete this section with the proofs of lemmata 1.16 and 1.17.

Proof of lemma 1.16 (order denseness)

Let $0 < f \in F$. Since $E^d = \{0\}$, there is an $e \in E^+$ with $e \wedge f > 0$. Using that F is Archimedean (as a consequence of its principal projection property), there is an $n \in \mathbb{N}$ such that $(e \wedge f - \frac{1}{n}e)^+ > 0$. Then $P^F_{(e\wedge f - \frac{1}{n}e)^+}(e)$ is a component of e in F, whence lies in E, and $\tilde{f} = \frac{1}{n}P^F_{(e\wedge f-\frac{1}{n}e)^+}(e)$ is a non-zero element in E that is $\leq f$.

◁ Indeed, denoting P^F with P temporarily, we have that $\tilde{f} \neq 0$, because

$$n\tilde{f} = P_{(e\wedge f-\frac{1}{n}e)^+}(e) \geq P_{(e\wedge f-\frac{1}{n}e)^+}((e \wedge f - \tfrac{1}{n}e)^+) = (e \wedge f - \tfrac{1}{n}e)^+ > 0\,.$$

Next, $\tilde{f} \leq f$, because $\tilde{f} \leq P_{(f-\frac{1}{n}e)^+}(\frac{1}{n}e)$, while the latter is $\leq f$ (for $u := f \geq 0$ and $v := \frac{1}{n}e \geq 0$ one has $P_{(u-v)^+}(v) = P_{(u-v)^+}(v - (v-u)^+) \leq P_{(u-v)^+}(u) \leq u$) (use 0.38)

▷ □

Proof of lemma 1.17 (Dominance)

We first make two preliminary observations.

i. For every $f \in F^+$, there is an $e \in E$ such that f lies in the band generated by e in F.

ii. if $u \in E^+$, $f \in F^+$, then $P^F_f(u) \in E$ i.e. F-components of an element E are in E again.

◁ From the assumption "if D is a disjoint system in E^+ that is majorized in F, then F-sup $D \in E$", we infer as in (0) of the proof of 1.15 that

$$u, e \in E^+ \quad \Longrightarrow \quad P^F_u(e) \text{ exists, and } P^F_u(e) \in E\,. \qquad (')$$

Now take $f \in F^+$. Choose a maximal disjoint system D in $\{u \in E : 0 \leq u \leq f\}$. Since D is a disjoint system in E majorized by $f \in F$, $e := F\text{-}\sup D \in E$ and $e \leq f$ By the latter fact on the one hand, and by the order denseness of E in F, ('), and the maximality of D on the other, we deduce

$$f_1 \perp e \quad \Longleftrightarrow \quad f_1 \perp f \qquad (f_1 \in F^+)\,.$$

Thus the band generated by e in F coincides with the band generated by f in F This establishes the first observation. The second observation then follows for $u \in E$ we have $P^F_f(u) = P^F_e(u) \in E$ by (').▷

Now let $f \in F^+$. We will find an $s \in E^+$ with $f \leq s$. First take an $e \in E^+$ such that f lies in the band generated by e in F. Applying 0.34 (with $u := f$, $\alpha_n := n$) we can construct a sequence f_n in F^+ such that $f_n \wedge f_m = 0$ if $|n - m| \geq 2$, $f = \sup_n f_n$, and $f_n \leq ne$ (all n). Set $g_n := nP^F_{f_n}(e) \geq P^F_{f_n}(f_n) = f_n$. Then $g_n \in E$, $0 \leq f_n \leq g_n$,

$g_n \le f + 2e$ (by 0.34), and $g_n \wedge g_m = 0$ if $|n - m| \ge 2$ (since $g_n \in B^F_{f_n}$). Since $(g_{2n})_n$ is a disjoint sequence in E that is majorized in F, $s_1 := F\text{-}\sup_n g_{2n} \in E$ exists, and likewise $s_2 := F\text{-}\sup_n g_{2n+1} \in E$. Then $f = F\text{-}\sup_n f_n \le s_1 \vee s_2 = F\text{-}\sup_n g_n \le f + 2e$. Thus, $s := s_1 \vee s_2$ is a majorant of f. □

Open problem 1.18
At the beginning of this section we observed that, for a (Hausdorff) locally solid Riesz space, the σ-Levi property implies the disjoint σ-Levi property and uniform completeness. One may wonder whether the converse holds:

Do the disjoint σ-Levi property and uniform completeness imply the σ-Levi property?

(If so, we can obtain theorem 1.14 as a direct consequence of theorem 1.3).

Remarks 1.19
Ad open problem 1.18 For σ-Fatou topologies, we can solve 1.18.
Assume that

(i) E is uniformly complete;

(ii) (E, τ) has the disjoint σ-Levi property i.e.

$$(e_n)_n \text{ disjoint in } E^+ \,,\ \textstyle\sum_n e_n\ \tau\text{-bounded} \quad \Longrightarrow \quad \sup_n e_n \in E$$

(iii) (E, τ) is σ-Fatou.

Then (E, τ) has the σ-Levi property.

Proof.
(I) E is σ-Dedekind complete: because E is uniformly complete and is conditionally σ-laterally complete, so has the principal projection property. Apply [LuZa, 42.5, p. 278]

(II) We may assume that E has a weak unit e: if $0 \le u_n \uparrow$ in E and $\{u_n\}_n$ is τ-bounded, then $e_n := u_n - P_{u_{n-1}}(u_n)$, $n \in \mathbb{N}$, are pairwise disjoint and $\sum_n e_n$ is τ-bounded ($\sum_1^k e_n \le u_k$), so that $e := \sup_n e_n$ exists. Further, $\{u_n\}_n$ is contained in the band generated by e. Finally, the band generated by e inherits the properties (i), (ii), and (iii) (with respect to the restriction topology) from E, because it is an ideal of E, σ-order closed, and regular in E respectively (and τ is generated by σ-Fatou pseudonorms).

We have $E \subset E^{\sigma u} \subset E^u \simeq C^\infty(S)$ where E^u is the universal completion of E (and S an extremally disconnected compact Hausdorff space), and $E^{\sigma u}$ is the σ-universal completion of E (see proof of Theorem 23.27 in [AlBu, p. 177]) which exists because E is σ-Dedekind complete and which can be identified as

$$(E^{\sigma u})^+ := \{w^* \in E^u : \exists\, 0 \le w_n \in E \uparrow w^*\}\,.$$

Since E is σ-Dedekind complete and super order dense in $E^{\sigma u}$, E is an ideal of $E^{\sigma u}$. Also, since $E \subset E^{\sigma u}$ order dense, e is a weak unit of $E^{\sigma u}$.

(III) Let $0 \le u_n \uparrow$ in E be τ-bounded. Then $\sup_n u_n \in E^{\sigma u}$
Fremlin's lemma 0.12 tells us that either $\sup_n u_n$ exists in $C^\infty(S)$, so in $E^{\sigma u}$ (namely if the set $\{s : \sup_n u_n(s) = \infty\}$ has empty interior), or there is a $w^* \in C^\infty(S)$ such that for all $0 \le w \le w^*$ and all $r \in (0, \infty)$: $w = \sup_n(w \wedge u_n/r)$. The latter possibility is excluded: *suppose there is a $0 < w \in E$ such that $w = \sup_n(w \wedge u_n/r)$ for all $r \in (0, \infty)$.* Choose a σ-Fatou pseudonorm ρ such that $\rho(w) := \delta > 0$. Since $\{u_n\}_n$ is

τ-bounded, there is an $r \in (0,\infty)$ such that $r[\rho < \delta/2] \supset \{u_n\}_n$. Then $\delta = \rho(w) = \sup_n \rho(w \wedge u_n/r) \le \sup_n \rho(u_n/r) \le \delta/2$. *Contradiction.*

(IV) Let $0 \le u_n \uparrow$ *in* E τ*-bounded. Then* $\sup_n u_n \in E$

By the above we have that $u := \sup_n u_n \in E^{\sigma u}$. By lemma 0.34 there exist disjoint $w_n \in E^{\sigma u}$ such that $u = \sup_n w_n$ and $w_n \le ne$. Since E is an ideal of $E^{\sigma u}$, $w_n \in E$. Further, $\sum_n w_n$ is τ-bounded.

$\lhd$Let U be a solid, σ-order closed 0-neighborhood. Since $\{u_k\}_k$ is τ-bounded, there is an $r \in (0,\infty)$ such that $\{u_k\}_k \subset rU$. Then for all n, k: $0 \le \sum_1^n w_i \wedge u_k \in rU$ (as U is solid), whence $\sum_1^n w_i = \sup_k(\sum_1^n w_i \wedge u_k) \in rU$ (as U is σ-order closed).$\rhd$

Conclusion: $u = \sup_n w_n \in E$ by the disjoint σ-Levi property.

Ad theorem 1.15 For each infinite cardinality κ we have an analogue of theorem 1.15. In order to formulate those analogues, we first fix some notation.
Let (E,τ) be a locally solid Riesz space, and let D be a disjoint subset of E. For every finite subset J of D, $s_J := \sum_{d\in J} d = \sup J$ exists. Directing the finite subsets of D by inclusion, we obtain a net $(s_J)_J$ which we denote by $\sum_{J\subset D}^{\text{finite}} d$. We say that $\sum_{J\subset D}^{\text{finite}} d$ is τ-bounded if the set $\{s_J\}_J$ is, and we write $\sum_{J\subset D}^{\text{finite}} d \xrightarrow{\tau} s$ if $s_J \xrightarrow{\tau} s$.

Theorem (An analogue of theorem 1.15 for cardinality κ)
Let (E,τ) be a Hausdorff locally solid Riesz space and let κ denote an infinite cardinality.
Suppose that

$$\left.\begin{array}{l} D \text{ disjoint in } E^+\,, \\ \operatorname{Card}(D) \le \kappa\,, \\ \sum_{J\subset D}^{\text{finite}} d \quad \tau\text{-bounded} \end{array}\right\} \Rightarrow \sum_{J\subset D}^{\text{finite}} d \xrightarrow{\tau} \sup D\,. \qquad \text{(strongly disjoint } \kappa\text{-Levi)}$$

Then the conclusions i. and ii. of theorem 1.15 hold.
Assume moreover that τ admits a zero-neighborhood basis of cardinality at most κ. Then instead of iii. we have that every majorized $A \subset E$ has a supremum which is at the same time the supremum of a subset of A of cardinality at most κ (which property we could call κ-super Dedekind completeness). Further, all the other conclusions of 1.15 hold.

As a special case, we mention that by taking κ the cardinality of E, we obtain:

Theorem
Let (E,τ) be a Hausdorff locally solid Riesz space.

$$\left.\begin{array}{l} D \text{ disjoint in } E^+\,, \\ \sum_{J\subset D}^{\text{finite}} d \quad \tau\text{-bounded} \end{array}\right\} \Longrightarrow \sum_{J\subset D}^{\text{finite}} d \xrightarrow{\tau} \sup D\,. \qquad \text{(strongly disjoint Levi)}$$

Then $\hat{E}$ is a Lebesgue space, E is order dense and majorizing in $\hat{E}$, components of elements of E in $\hat{E}$ are actually in E, and E is topologically complete if and only if it is uniformly complete.

For a proof we only need to adapt the proof of theorem 1.15 at some points:
(0) The strongly disjoint κ-Levi property implies the strongly disjoint σ-Levi property, so $(*)$ and its conclusions in (0) continue to hold, except for (%): the fact that τ is Lebesgue and has a zero-neighborhood base of cardinality at most κ implies now that $\hat{E}$ is a κ-super Dedekind complete Riesz space.

(I) Instead of ($\flat$) we now have

$$U \subset C_E(e)\,,\ \mathrm{Card}(U) \le \kappa \quad \Longrightarrow \quad \hat{E}\text{-}\sup U\,,\ \hat{E}\text{-}\inf U \in E\,.$$

For the proof of this, we first choose a well-ordering on U: say $(\mathbb{A}, \prec)$ is a well-ordering and $\alpha \to u_\alpha$ is a bijection from $\mathbb{A}$ onto $U = \{u_\alpha\}_\alpha$. Next, we define disjoint $\{e_\alpha\}_\alpha$ by $e_\alpha := \sup_{\beta \preceq \alpha} u_\beta - \sup_{\beta \prec \alpha} u_\beta$ and show as before, but now with transfinite induction, that

$$P(\alpha): \begin{cases} e_\alpha \in E \quad (\text{so } e_\alpha \in C_E(e)\ ); \\ \hat{E}\text{-}\sup_{\beta \preceq \alpha} e_\beta = \hat{E}\text{-}\sup_{\beta \preceq \alpha} u_\beta\,, \end{cases}$$

holds for all α. In combination with the κ-super Dedekind completeness, we obtain from this again (&).

$\lhd$The transfinite induction proof proceeds as follows. Suppose $\alpha \in \mathbb{A}$ and $\forall_{\beta \prec \alpha} P(\beta)$. A first consequence of the latter is that $\{e_\beta : \beta \prec \alpha\}$ is a disjoint system *in* $C_E(e)$, whence $\hat{E}\text{-}\sup_{\beta \prec \alpha} e_\beta = E\text{-}\sup_{\beta \prec \alpha} e_\beta \in E$ by virtue of the strongly disjoint κ-Levi property. A second consequence is that $\hat{E}\text{-}\sup_{\beta \prec \alpha} u_\beta = \hat{E}\text{-}\sup_{\beta \prec \alpha} e_\beta$. Combining:

$$e_\alpha := \underbrace{\underbrace{u_\alpha}_{\in E} \vee \hat{E}\text{-}\sup_{\beta \prec \alpha} u_\beta}_{\in E} \;-\; \underbrace{\hat{E}\text{-}\sup_{\beta \prec \alpha} u_\beta}_{\in E} \;\in E\,.$$

Since $e_\alpha \perp \{e_\beta : \beta \prec \alpha\}$:

$$e_\alpha \vee \sup_{\beta \prec \alpha} e_\beta = e_\alpha + \sup_{\beta \prec \alpha} e_\beta = (\sup_{\beta \preceq \alpha} u_\beta - \sup_{\beta \prec \alpha} u_\beta) + \sup_{\beta \prec \alpha} u_\beta = \sup_{\beta \preceq \alpha} u_\beta,$$

which establishes the second half of $P(\alpha)$.$\rhd$

(III) The property ($\sharp$) now follows from the κ-super Dedekind completeness of $\hat{E}$ and the disjoint κ-Levi property of E.

Ad lemma 1.16 In the order denseness lemma 1.16 we can weaken the last two assumptions. Instead of the principal projection property it suffices that F has sufficiently many projections. Further, not all components of elements of E in F have to be in E, but "sufficiently many" in the sense that if $f \in F^+$ is a non-zero component of $e \in E^+$, then there exists a component e' of e in E such that $0 < e' \le f$.

Indeed, given $f \in F^+$ we can find (as in the proof of lemma 1.16) an $e \in E^+$ and $n \in \mathbb{N}$ such that $\tilde{e} := (e \wedge f - \frac{1}{n}e)^+ > 0$. Choose now a projection band B inside the band generated by $\tilde{e}$. Then $0 < n^{-1}P_B(e) \le f$. Next choose a component u of e in E with $0 < u \le n^{-1}P_B(e)$.

1.4 Ando's characterization of L^p-spaces and c_0-spaces in terms of contractive projections

A (positive) contractive projection in a quasi-Banach lattice $(E, \|\|)$ is a linear map $P : E \to E$ such that for all $x \in E$: $(P(x) \ge 0$ if $x \ge 0,)$ $\|P(x)\| \le \|x\|$, and $P^2(x) = P(x)$. In [An66] T. Ando proved

Theorem 1.20
Let E be a Banach lattice of dimension at least 3.
Then E is isometrically Riesz isomorphic to an L^p-space for some $p \in [1, \infty)$ or to a $c_0(X)$ for some discrete set X if and only if each closed Riesz subspace of E is the range of a positive contractive projection.

The question we want to discuss here is: can we drop the convexity condition in 1.20 i.e.

can we replace "Banach lattice" by "quasi-Banach lattice" to include L^p-spaces for $p \in (0, 1)$ in the characterization? (∗)

The answer follows from another theorem of Ando ([An69]):

Theorem 1.21
Let $(S, \mathcal{A}, \mu)$ be a finite measure space, let $p \in (0, 1)$, and let E be $L^p(\mu)$ or $L^p_{\mathbb{C}}(\mu)$. For a measurable set $B \in \mathcal{A}$ let $P_B : E \to E$ be the mapping defined by

$$P_B(f) = f \cdot \mathbb{1}_B \quad (f \in E) .$$

Then $P : E \to E$ is a contractive projection if and only if there exists a measurable set $B \in \mathcal{A}$ and a contraction $V : E \to E$ such that

$$P = P_B + V \ , \ P_B V = V \ , \ V P_B = 0 \, .$$

In particular, for $p \in (0, 1)$, the range of a contractive projection is a band, which excludes the closed Riesz subspace of the even functions of $L^p[-1, 1]$ from being the range of a contractive projection. Therefore the answer to (∗) is no.

In the remainder of this section we will present an alternative proof of 1.21, which yields a slightly more general result (theorem 1.22).
The argument we will present is different from Ando's original in two respects. First, it gives a prominent rôle to the elementary observation that for $p \in (0, 1)$ and $s, t \in [0, \infty)$ we have $(s + t)^p \le s^p + t^p$ with equality if and only if $s \wedge t = 0$ (which yields lemma 1.23); secondly, it is phrased in terms of the Riesz space structure of L^p.
Those two features allow us to drop the finiteness condition on the measure, while obtaining a slightly more general result:

Theorem 1.22
Let $(S, \mathcal{A}, \mu)$ be a measure space and let $p \in (0, 1)$.
Let E be the real Riesz space $L^p(\mu)$ (or the complex Riesz space $L^p_{\mathbb{C}}(\mu)$, see p. 27). For a band B (complex band, respectively) of E denote the projection onto B by P_B. Then $P : E \to E$ is a contractive projection if and only if there exists a band B of E and a contraction $V : E \to E$ such that

$$P = P_B + V \, , \ P_B V = V \, , \ V P_B = 0 \, ,$$

where P_B denotes the band projection onto B

That theorem 1.22 is indeed a generalization of theorem 1.21 follows from a characterization of bands in $L^p(\mu)$ if μ is (σ-)finite: in that case every band B in L^p is induced by a measurable set S_B as

$$B = \{g \in L^p : \ g\mathbb{1}_{S_B} = g\} \quad \text{(see 0.22)}.$$

Before turning to the proof of 1.22, we make two observations. The first crucial observation is that we have a converse to the p-additivity of $\|\ \|_p$ in $L^p(\mu)$ as well as in $L^p_{\mathbb{C}}(\mu)$ for $p \in (0,1)$.

Lemma 1.23 (converted p-additivity)
Let $(S, \mathcal{A}, \mu)$ be a measure space and let $p \in (0,1)$. Then

$$\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p \quad \Longleftrightarrow \quad |f| \wedge |g| = 0 \qquad (f, g \in L^p_{\mathbb{C}}(\mu)).$$

Proof
First observe that

$$\text{for } s,t \in [0,\infty) : \quad \begin{cases} (s+t)^p \le s^p + t^p \\ (s+t)^p = s^p + t^p \quad \Longleftrightarrow \quad s \wedge t = 0 \end{cases} \tag{!}$$

◁We only need to consider the non-trivial case: both $s > 0$ and $t > 0$.
Using that $r \le r^p$ $(r \in [0,1])$ for $p < 1$, we have

$$1 = \left(\frac{s}{s+t}\right) + \left(\frac{t}{s+t}\right) \le \left(\frac{s}{s+t}\right)^p + \left(\frac{t}{s+t}\right)^p, \tag{\%}$$

which yields the upper part of (!).
Further, if $(s+t)^p = s^p + t^p$, then the right hand side of (%) equals 1 i.e.

$$\frac{s}{s+t}, \frac{t}{s+t} \in \{r \in [0,1] : r = r^p\} = \{0,1\}.$$

Hence $s \wedge t = 0$.▷

Now let $f, g \in L^p_{\mathbb{C}}$ with $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p$. Then

$$\begin{aligned} \|\,|f|\,\|_p^p + \|\,|g|\,\|_p^p = \|f\|_p^p + \|g\|_p^p = \|f+g\|_p^p &\le \|\,|f|+|g|\,\|_p^p \\ &\le \|\,|f|\,\|_p^p + \|\,|g|\,\|_p^p \end{aligned}$$

i.e. $\|\,|f|+|g|\,\|_p^p = \|\,|f|\,\|_p^p + \|\,|g|\,\|_p^p$. From that and the upper part of (!) we see that $|f|^p + |g|^p - (|f|+|g|)^p$ is a positive function whose integral vanishes.
Then $|f(s)| \wedge |g(s)| = 0$ for μ-almost all $s \in S$ by the lower part of (!). □

Our second observation addresses itself to the case that we have two disjoint decompositions of an element and we want to determine if they are not in fact the same.

Lemma 1.24
Let E be a Riesz space and let $x, y, x', y' \in E$ such that $x + y = x' + y'$. Suppose $|x| \wedge |y| = 0$, $|x| \wedge |y'| = 0$, $|x'| \wedge |y'| = 0$ and $|x'| \wedge |y| = 0$.
Then $x = x'$ and $y = y'$.
Corollary: the same lemma holds for $E = L^p_{\mathbb{C}}$.

◁Observing that

$$\begin{aligned} |x - x'| = |y' - y| &= |x - x'| \wedge |y' - y| \\ &\le |x| \wedge |y'| + |x| \wedge |y| + |x'| \wedge |y'| + |x'| \wedge |y| = 0, \end{aligned}$$

we see that $x = x'$ and $y = y'$.
For elements of $L^p_{\mathbb{C}}$ apply the lemma to the real and imaginary parts successively.▷

We now turn to the proof of theorem 1.22

Proof of 1.22

We prove the *only if*-part in five steps.

Suppose $P : E \to E$ is a contractive projection with range B.

The following two statements apply to both $E = L^p$ and $E = L^p_{\mathbb{C}}$.

(I) $B = \mathrm{Ker}(P^2 - P)$ is a closed linear subspace of E.

(II) Components of elements of B in E lie in B i.e. if $f, g \in E$ such that $f + g \in B$ and $|f| \wedge |g| = 0$, then $f, g \in B$.

$\triangleleft$Let $f, g \in E$, $|f| \wedge |g| = 0$ such that $f + g \in B$. Then $f + g = P(f + g)$, so

$$\begin{aligned}\|f+g\|_p^p &= \|P(f+g)\|_p^p = \|Pf+Pg\|_p^p \leq \|Pf\|_p^p + \|Pg\|_p^p \\ &\leq \|f\|_p^p + \|g\|_p^p = \|f+g\|_p^p ,\end{aligned}$$

so $\|Pf + Pg\|_p^p = \|Pf\|_p^p + \|Pg\|_p^p$ and by lemma 1 23, $|Pf| \wedge |Pg| = 0$

Further, we also see the following chain of inequalities uncoil

$$\begin{aligned}\|f+g\|_p^p &= \|Pf+Pg\|_p^p = \|P(Pf+g)\|_p^p \leq \|Pf+g\|_p^p \\ &\leq \|Pf\|_p^p + \|g\|_p^p \leq \|f\|_p^p + \|g\|_p^p = \|f+g\|_p^p ,\end{aligned}$$

which means $\|Pf + g\|_p^p = \|Pf\|_p^p + \|g\|_p^p$, so $|Pf| \wedge |g| = 0$ by 1 23. A similar argument shows $|f| \wedge |Pg| = 0$

But then we are in the position to apply 1.24 to $f + g = Pf + Pg$, which yields the result announced $\triangleright$

(III) Suppose $0 \leq v \in L^p$, $0 \leq u \in B \cap L^p$, $0 \leq v \leq u$. Then $v \in B \cap L^p$.

$\triangleleft$Indeed, since L^p is weak-Freudenthal, there exists a sequence $(v_n)_n$ of linear combinations of components of u that converges u-uniformly to v. By (II), this sequence lies in B and so does $v = \| \, \|_p - \lim_n v_n$.$\triangleright$

(IV) B is a band.

Case $E = L^p$. If $x \in B \subset L^p$, then x^+ and $-x^-$ are components of x in L^p, and, by (II), they are in B. Thus, B is a Riesz subspace. But then (III) implies that B is even an ideal in L^p. Therefore, B is a closed ideal of L^p, hence by the Lebesgue property of L^p automatically a band.

Case $E = L^p_{\mathbb{C}}$. To stress that the range of P is now a complex-linear subspace, we momentarily denote it by $B_{\mathbb{C}}$. Our first observation is that

$$f \in B_{\mathbb{C}} \Rightarrow |f| \in B_{\mathbb{C}} .$$

$\triangleleft$The function $\mathfrak{z} : \mathbb{C} \to \{z \in \mathbb{C} : |z| = 1\}$ defined by

$$\mathfrak{z}(z) = \lim_n \frac{\bar{z}}{|z| + \frac{1}{n}} = \begin{cases} |z|/z \quad (= \bar{z}/|z|) & \text{if } z \neq 0 , \\ 0 & \text{if } z = 0 , \end{cases}$$

is bounded and Borel($\mathbb{C}$)-measurable. Therefore there exist a sequence of bounded Borel($\mathbb{C}$)-stepfunctions $(h_n)_n$, such that $|\mathfrak{z}(z) - h_n(z)| \leq 1/n \quad (z \in \mathbb{C},\ n \in \mathbb{N})$.

Let now $f \in B_{\mathbb{C}}$. Then each $h_n(f) \cdot f$ is a linear combination of components of f and hence lies in $B_{\mathbb{C}}$.

By Lebesgue's dominated convergence theorem, $|f| = \| \, \|_p$-$\lim_n h_n(f) \cdot f \in B_{\mathbb{C}}$.$\triangleright$

Next, if we set $B := B_{\mathbb{C}} \cap L^p$, and apply (III) to $v = \mathrm{Re}(f)^{\pm}, \mathrm{Im}(f)^{\pm}$ and take $u = |f| \in B \cap L^p$ we obtain that $B_{\mathbb{C}} = B + iB$.
Finally, since both $B_{\mathbb{C}}$ and L^p are closed, so is B, and we can use the same arguments as in the case $E = L^p$, to show that B is a band of L^p. Therefore, $B_{\mathbb{C}}$ is a complex band in $L^p_{\mathbb{C}}$ (see section 0.3.13).

(V) $P - P_B := V$ is a contraction with $VP_B = 0$, $P_B V = V$, and $P = P_B + V$.
For $x \in B$: $P(x) = x = P_B(x)$ so $V(x) = 0$, while for $x \in B^d$: $V(x) = P(x) \in D$. Consequently, $VP_B = 0$ and $P_B V = V$, and from this we get that

$$V = V \circ (P_B + P_{B^d}) = V \circ P_{B^d} = (P - P_B) \circ P_{B^d} = P \circ P_{B^d} \tag{\#}$$

is a contraction.
We conclude with the proof of the *if-part* of the theorem.
If B is a band and $P = P_B + V$ with P_B and V as prescribed, then it follows as in (#) that P is a contraction and also that

$$P^2 = P_B^2 + P_B V + VP_B + V^2 = P_B^2 + V + 0 + (V(P_{B^d} + P_B)) \cdot (P_B V) = P_B + V + 0 + 0\,.$$

□

Remarks 1.25

About Ando's proof of theorem 1.21 Ando originally formulated and proved theorem 1.21 for $E = L^p_{\mathbb{C}}(\mu)$ with μ finite: however, the proof he gave can be taken over literally for the case $E = L^p(\mu)$. More essential is his use of the (σ-)finiteness of μ: from that he proves that one can find a function with *maximal support* in the range of a contractive projection i.e. the range of a contractive projection (which is a Riesz space in view of theorem 1.22) has a weak unit. In case μ is only locally finite, this may prove impossible: take $S = \mathbb{R}$, $\mathcal{A}$ the collection of all countable and co-countable subsets of $\mathbb{R}$ and μ the counting measure. Then every $f \in L^p(\mu)$ has countable support, and so

$$D := \{f \in L^p \;:\; f = 0 \text{ on } [0,\infty)\} = \{\mathbb{1}_{\{t\}} \;:\; t \in [0,\infty)\}^{\perp}$$

is a band without a function with maximal support.

About the implication of theorem 1.21 by theorem 1.22 In the (σ-)finite case, every band B in L^p is induced by a measurable set S_B, as

$$B = \{g \in L^p : g\mathbb{1}_{S_B} = g\}\,,$$

(see e.g. [dJvR, 4I, p. 30]) and so theorem 1.22 implies theorem 1.21.
In the locally finite case this need not be the case however: in the example above, D is a band that does not correspond to a measurable set.

About an analogue of Ando's characterization for $p \in (0,1)$ Given that we cannot *include* L^p-spaces for $p \in (0,1)$ in theorem 1.20, the question remains whether theorem 1.20 has an analogue which characterizes L^p-spaces for $p \in (0,1)$.
A crucial point in Ando's proof for 1.20 is

Let E be a Banach lattice such that
if D is a 3-dimensional Riesz subspace of E and D_0 a 2-dimensional Riesz subspace of D, then there exists a positive contractive projection from D onto D_0.
Then the norm of E is p-additive for some $p \in [1,\infty]$, and hence E is isometrically Riesz isomorphic to an L^p-space, for some $p \in [1,\infty)$, or to a closed Riesz subspace of a $C(S)$ with S compact Hausdorff. (∗)

Translating the concepts and arguments using convexity in Ando's proof of $(*)$ into their concave counterparts, one can obtain:

> *Let $(E, |\!|\!|\ |\!|\!|)$ be a quasi-Banach lattice with $|\!|\!|\ |\!|\!|$ super-additive on the positive cone (i.e. $|\!|\!|u|\!|\!| + |\!|\!|v|\!|\!| \leq |\!|\!|u+v|\!|\!|$ for $u, v \in E^+$) such that*
>
> > *if D is a 3-dimensional Riesz subspace of E and D_0 a 2-dimensional Riesz subspace of D, then there exists a positive projection P from D onto D_0 that is dilative on the positive cone (i.e. $|\!|\!|P(u)|\!|\!| \geq |\!|\!|u|\!|\!|$ for $u \in E^+$).*
>
> *Then the quasi-norm of E is p-additive for some $p \in (0,1]$, and hence E is isometrically Riesz isomorphic to an L^p-space, for some $p \in (0,1]$.* (#)

For a non-prejudiced comparison of (#) with $(*)$ observe that a positive projection in a normed Riesz space is contractive if and only if it is contractive on the positive cone, and that Riesz quasi-norm is a norm if and only if it is subadditive on the positive cone. In other words, $(*)$ expresses a property of the positive cone.
Since every closed Riesz subspace of L^p, $p \in [1, \infty)$, or $c_0(X)$ is the range of a positive contractive projection, and since $c_0(X)$ is the only type of M-space that provides positive contractive projections onto each of its closed Riesz subspaces [LiTz2, proof of thm 1.b.8], theorem 1.20 is a consequence of $(*)$. However, in L^p-spaces with $p \in (0,1)$, not every closed Riesz subspace is the range of a positive projection that is dilative on the positive cone: band projections will be contractive.

1.5 Characterization of band projections among the contractive projections in L^p for $p \in (0,1)$

Let $p \in (0,1)$ and let P be a contractive projection in L^p with range B. Ando's characterization of contractive projections in L^p (1.22) tells us that $P = P_B + V$, where P_B is the band projection onto B, and V is a contraction that maps $B^\perp$ into B and B to $\{0\}$.
In this section we are interested in the question under what condition P coincides with P_B. As an answer, we will describe two necessary and sufficient conditions on P to be purely the band projection onto B.
Our first approach is to observe that P_B is an order bounded map $L^p \to L^p$ (therefore having an absolute value) and to view the contraction $V = P - P_B$ as a perturbation that enlarges the absolute value. For this approach to make sense we must know

Lemma 1.26
Let $T : L^p \to L^p$ be a continuous linear map. Then T is order bounded.

Proof
Let $e \in L^{p+}$. Using the Levi property of $\|\ \|_p$, we show that there exists a $c \in L^{p+}$ such that $T[0,e] \subset [-c, c]$.
Call $P \subset E^+$ a *disjoint partition* of e if P is a finite set of components of e with supremum (hence sum) e. The set $\mathcal{P}$ of all disjoint partitions of e is directed by refinement: $P \prec P'$ (P' refines P) if P' is a union of disjoint partitions of the elements of P.

Define a mapping $\mathcal{P} \to L^p$ by

$$T[P] := \sum_{u \in P} |T(u)| .$$

Then we obtain an increasing net $(T[P])_{P \in \mathcal{P}}$ that is bounded in $\| \ \|_p$:

$$\begin{aligned} \| T[P] \|_p^p &= \| \textstyle\sum_{u \in P} |T(u)| \|_p^p \le \sum_{u \in P} \| |T(u)| \|_p^p \le [\![T]\!]^p \sum_{u \in P} \| (u) \|_p^p \\ &= [\![T]\!]^p \| \textstyle\sum_{u \in P} u \|_p^p = [\![T]\!]^p \| e \|_p^p , \end{aligned}$$

and therefore order bounded by the Levi-property (0.4). Thus there is a $c \in L^{p+}$ such that

$$\sum_1^n |T(e_\iota)| \le c \quad \text{whenever } \{e_\iota\}_1^n \text{ is a disjoint partition of } e . \qquad (*)$$

Now let $0 \le a \le e$. Since L^p is weak-Freudenthal, there exists a sequence a_n in

$$\{\textstyle\sum_1^n \alpha_\iota e_\iota : \ \{e_\iota\}_1^n \text{ disjoint partition of } e, \ \alpha_\iota \in [0,1]\}$$

with $\|a - a_n\|_p \to 0$ (0.66). By $(*)$ and 0.123: $|T(a)| = \| \ \|_p\text{-}\lim_n |T(a_n)| \le c$. □

The above lemma implies that $V := P - P_B$ is order bounded. Using the formulas (0.84) we successively check that $|V| = 0$ on B, and that $|V| \wedge P_B = 0$ (for the latter observe that $0 \le P_B \wedge |V|(u) \le P_B(P_{B^\perp}(u)) + |V|(P_B(u)) = 0$ for all $u \in E^+$). Therefore: $|P_B + V| = P_B + |V| > P_B$ if $P \ne P_B$, which yields the following:

Corollary 1.27
Let P be a contractive projection with range B.
Then P is the band projection onto B if and only if $|P| \le |Q|$ for all contractive projections Q with range B.

The second approach to characterize band projections is to observe that if V is a contraction with $P_B V = V$ and $V P_B = 0$, then both $P_B + V$ and $P_B - V$ are contractive projections with the range B. So P_B is in a sense the center of all contractive projections with range B.

Definition 1.28
For a vector space W and a subset $A \subset W$ we call $c \in A$ a center of A if for all $w \in W$: $c - w \in A$ whenever $c + w \in A$.

A set contains either 0, 1, or infinitely many centers. In the latter case, the centers lie "cofinally" on one dimensional varieties:

Lemma 1.29
Let W be a vector space, let $A \subset W$, and let c, c' be centers of A.
Then $c \pm n(c' - c) \in A$ for all $n \in \mathbb{N}_0$.

◁For $n = 0, 1$ this is clear. Further, if $c - n(c' - c) = c' - (n+1)(c' - c) \in A$ for some $n \in \mathbb{N}_0$, then $A \ni c' + (n+1)(c' - c) = c + (n+2)(c' - c)$, as c' is a center.▷

Since the set of contractions is bounded in the quasi-normed space of all continuous linear transformations of L^p, we obtain the following corollary.

Corollary 1.30
The band projection P_B onto B is the unique center of all contractive projections in L^p with range B.

An intriguing question remains:

Open problem 1.31
Can we give a satisfactory description of contractions V with $P_B V = V$ and $VP_B = 0$?

Remarks 1.32
About linear isometries in L^p for $p \in (0,1)$
The ideas used in the previous two sections give us tools to reveal something of the structure of linear isometries in L^p for $p \in (0,1)$.
Let $p \in (0,1)$ and let $T : L^p \to L^p$ be a linear isometry. By lemma 1.26 and lemma 1.23 respectively, T is order bounded and disjointness preserving.
As can be seen from the formulas 0.84 the disjointness preserving character of T passes on to its absolute value, and a fortiori to its positive and negative parts, which implies that $|T|$, T^+ and T^- are all Riesz homomorphisms.
A result of Meyer on order bounded, disjointness preserving operators ([M-N, theorem 3.1.4, p. 150]) tells us that

$$|T|(u) = |T(u)|\ ,\ T^+(u) = (Tu)^+\ ,\ T^-(u) = (Tu)^- \qquad (u \in L^{p+})\ .$$

As a result of that and the fact that $T, |T|$ are disjointness preserving, $|T|$ is an isometry: if $x \in L^p$, then

$$\begin{aligned}\|\,|T|x\,\|_p^p &= \|\,|T|x^+\,\|_p^p + \|\,|T|x^-\,\|_p^p = \|\,|Tx^+|\,\|_p^p + \|\,|Tx^-|\,\|_p^p \\ &= \|\,Tx^+\,\|_p^p + \|\,Tx^-\,\|_p^p = \|\,Tx\,\|_p^p = \|\,x\,\|_p^p\ .\end{aligned}$$

Since a positive linear continuous map from a Lebesgue topological Riesz space to a Hausdorff locally solid Riesz space is automatically order continuous (0.46, 0.126 iii), we obtain that $|T|$ is an isometric, order continuous Riesz isomorphism. Hence, $|T|(L^p)$ is an isomorphic copy of L^p Moreover, $T^\pm$ are order continuous: $0 \le T^\pm(u_\alpha) \le |T|(u_\alpha) \downarrow 0$ if $0 \le u_\alpha \downarrow 0$ (cf. 0.46).
As for T^+ and T^-, we can observe that their ranges are disjoint Riesz subspaces of $|T|(L^p)$ ($\simeq L^p$):

Let $u, v \in L^{p+}$. Then $T^+(u) \wedge T^-(v) = 0$.

$\lhd$ Indeed, if $v \in B_u^d$, then $T^+(u) \wedge T^-(v) \le |T|(u) \wedge |T|(v) = 0$ because $|T|$ is disjointness preserving, while if $v \in B_u$, then

$$\begin{aligned}T^+(u) \wedge T^-(v) &= T^+(u) \wedge T^-(\sup_n v \wedge nu) = T^+(u) \wedge \sup_n T^-(v \wedge nu) \\ &\le \sup_n (T^+(u) \wedge T^-(nu)) \le \sup_n n(T(u)^+ \wedge T(u)^-) = 0\end{aligned}$$

by using the formulas of Meyer and the order continuity of T^-. $\rhd$

Letting P_{T^+}, P_{T} be the projections in $|T|(L^p)$ onto the (disjoint) bands generated by $T^+(L^p)$ and $T^-(L^{p+})$ in $|T|(L^p)$ respectively, we obtain that

$$T^+ = P_{T^+} \circ |T|\ , \quad T^- = P_{T^-} \circ |T|\ .$$

Chapter 2

Riesz homeomorphic characterizations

In this chapter we set out to generalize a joint characterization of L^p-spaces with $p \in [1, \infty)$, and order continuous M-spaces [LiTz2, theorem 1.b.13 p. 22]. Basically, the result we obtain is the following (cf. theorem 2.34 on page 89).

Theorem

Let $(E, |\!|\!|\;|\!|\!|)$ be a quasi-Banach lattice with the property:

$$\left.\begin{array}{l}\text{if } (x_n)_n \text{ and } (y_n)_n \text{ are two disjoint sequences with } |\!|\!| x_n |\!|\!| = |\!|\!| y_n |\!|\!|, \\ \text{then } \sum_1^\infty x_n \text{ exists if and only if } \sum_1^\infty y_n \text{ exists.}\end{array}\right] (\pitchfork)$$

Then there exists a $p \in (0, \infty]$ and a quasi-norm $|\!|\!|\;|\!|\!|_{(p)}$ on E such that $|\!|\!|\;|\!|\!|_{(p)}$ is p-additive.

Above, and in the rest of this chapter, the following conventions are used:

Convention 2.1

Given $p \in (0, \infty]$, by saying that $|\!|\!|\;|\!|\!| : E \to [0, \infty)$ is p-additive, we mean:

$$|\!|\!| u + v |\!|\!| = \begin{cases} [\, |\!|\!| u |\!|\!|^p + |\!|\!| v |\!|\!|^p \,]^{1/p} & \text{if } p < \infty\,, \\ |\!|\!| u |\!|\!| \vee |\!|\!| v |\!|\!| & \text{if } p = \infty \end{cases} \qquad (u, v \in E^+\,,\ u \wedge v = 0)\,,$$

and we use the following notation

$$\ell^p \cap c_0 := \begin{cases} \text{the quasi-Banach space } (\ell^p, \|\;\|_p) & \text{if } p < \infty\,, \\ \text{the Banach space } (c_0, \|\;\|_\infty) & \text{if } p = \infty\,. \end{cases}$$

The strategy we use to prove the theorem above is by and large the same as that for [LiTz2, theorem 1.b.13, p. 22], but its elaboration differs: amongst other things, because we have to bypass the use of convexity. More specifically, our strategy is reflected in the structure of this chapter as follows.
After a section to introduce the relevant theory on bases (section 2.1) we generalize Zippin's theorem on perfectly homogeneous bases in section 2.2.
With help of Zippin's theorem and some theory on quasi-normed Riesz spaces (developed in section 2.3) we see in section 2.4 that $(\pitchfork)$ implies that there exists a $p \in (0, \infty]$ such that for every disjoint sequence $(x_n)_n$ in E: $\sum_1^\infty x_n$ exists if and only if $(|\!|\!| x_n |\!|\!|)_n \in \ell^p \cap c_0$. In fact, we strengthen the latter, and show that there exists an uniform constant $M \in (0, \infty)$ such that

$$M^{-1} \left\| (|\!|\!| x(n) |\!|\!|)_n \right\|_p \le \left|\!\left|\!\left| \textstyle\sum_1^N x_n \right|\!\right|\!\right| \le M \left\| (|\!|\!| x(n) |\!|\!|)_n \right\|_p \qquad (\between)$$

for every disjoint (finite or infinite) sequence $(x_n)_n$ in E.

If $p \in (0, \infty)$, we define $|\!|\!|\;|\!|\!|_\triangledown$ and $|\!|\!|\;|\!|\!|_\vartriangle$ by

$$|\!|\!| x |\!|\!|_\triangledown := \inf \left\{ [\sum_1^n |\!|\!| x_\iota |\!|\!|^p]^{1/p} \ :\ n \in \mathbb{N},\ \sum_1^n x_\iota = x \text{ disjoint} \right\} \qquad (x \in E)$$

and $|\!|\!| x |\!|\!|_\vartriangle$ by the expression we obtain from the above one by replacing the infimum by a supremum. Then $|\!|\!|\;|\!|\!|_{(p)} := |\!|\!|\;|\!|\!|_{\vartriangle\triangledown}$ defines a p-additive Riesz quasi-norm on E that is equivalent to $|\!|\!|\;|\!|\!|$ by $(\between)$, and theorem 1.2 yields that $(E, |\!|\!|\;|\!|\!|_{(p)})$ is isometrically Riesz isomomorphic to an L^p-space.

If $p = \infty$,

$$|\!|\!| x |\!|\!|_{(\infty)} := \inf \{ \textstyle\bigvee_1^n |\!|\!| x_\iota |\!|\!| \ :\ n \in \mathbb{N},\ \bigvee_1^n x_\iota = x \text{ disjoint} \} \qquad (x \in E)\,,$$

defines an equivalent ∞-additive Riesz quasi-norm on E. Under some extra conditions which are studied in section 2.5 we can show that $(E, |\!|\!|\;|\!|\!|_{(\infty)})$ is isometrically Riesz isomorphic to an M-space.

2.1 Unconditional bases in quasi-Banach spaces

In the next section we will generalize Zippin's theorem on perfectly homogeneous bases from the context of Banach spaces to that of quasi-Banach spaces. To prepare us for that, this section will carry over the relevant concepts and results from the Banach space setting to the quasi-Banach space setting.

We start with translating the notion of a basis.

Definition 2.2

Let $(E, \rrbracket \llbracket)$ be a quasi-Banach space.

A sequence $(e_n)_1^\infty$ in E is called a Schauder basis (or briefly, basis) of E if for every $x \in E$ there exists a unique sequence of scalars $(x(n))_1^\infty$ such that

$$x = \rrbracket \llbracket \text{-}\lim_N \textstyle\sum_1^N x(n) e_n .$$

Let $(e_n)_1^\infty$ be a Schauder basis of E.

If $x = \sum_1^\infty x(n) e_n$, we call the $x(n)$, $n \in \mathbb{N}$, the (expansion) coefficients of x (with respect to the basis $(e_n)_1^\infty$).

By the uniqueness of the expansion coefficients, the maps

$$P_n : E \to E \,,\ \textstyle\sum_1^\infty x(n) e_n \mapsto x(n) e_n \quad (n \in \mathbb{N}) ,$$

are well-defined and linear. These maps are referred to as the basic projections (of the basis $(e_n)_1^\infty$).

A sequence $(a_n)_1^\infty$ in E that is a Schauder basis of its closed linear span, is called a basic sequence.

Example 2.3

The sequence

$$e_n := (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, 0, \ldots) \quad (n \in \mathbb{N})$$

is a normalized Schauder basis of $(c_0, \| \, \|_\infty)$, and of $(\ell^p, \| \, \|_p)$, $0 < p < \infty$. This sequence is referred to as the standard basis of c_0 or ℓ^p respectively.

A glimpse of the power of the concept of a basis is revealed by

Lemma 2.4

Let $(E, \rrbracket \llbracket)$ be a quasi-Banach space equipped with a Schauder basis $(e_n)_1^\infty$ and let $(P_n)_1^\infty$ be the associated sequence of basic projections.

Set $S_N^M := P_N + \ldots + P_M$ ($N \le M$ in $\mathbb{N}$).

Then the collection of operators, S_N^M, $N \le M$ in $\mathbb{N}$, is uniformly continuous, in fact there is a $C \in (0, \infty)$ such that

$$\rrbracket \sum_N^M x(n) e_n \llbracket \ \le C \rrbracket \sum_{N'}^{M'} x(n) e_n \llbracket \quad \left(\begin{array}{l} \text{if } x(n), n \in \mathbb{N}, \text{ scalars, and} \\ N' \le N \le M \le M' \text{ in } \mathbb{N} \end{array} \right) . \tag{$\mho$}$$

In particular, the basic projections are (uniformly) continuous.

Most results in this section with exception of 2.10 and 2.14 are proven in essentially the same way as their Banach space analogues, the reason for this being that the uniform boundedness principle, open mapping theorem, and closed graph theorem hold in a quasi-normed space (by virtue of its metrizability as topological vector space ([Kö, §15: 13(2), 12(1) and 12(3) respectively]).

◁As an example we prove 2.4. Observing that it suffices to prove ($\mho$) for an equivalent quasi-norm, we will assume that $\|\ \|$ is r-subadditive and construct an equivalent quasi-norm $\|\ \|'$ that satisfies ($\mho$).

For each $x \in E$ the series $\sum_n x(n)e_n$ is $\|\ \|$-bounded, so the expression

$$\|x\|' := \sup\left\{ \|\textstyle\sum_N^M x(n)e_n\| \ : \ N < M \text{ in } \mathbb{N}\right\}$$

is finite. From its definition, $\|\ \| \le \|\ \|'$ and $\|\ \|'$ is an r-subadditive quasi-norm (0.97). Now, $\|\ \|'$ satisfies ($\mho$) with $C = 1$, so we are done if $\|\ \|'$ is equivalent with $\|\ \|$. To this end, we prove E is complete with respect to $\|\ \|'$.

◁Indeed, if $(x_\iota)_1^\infty$ is a $\|\ \|'$-Cauchy sequence in E, then there exist $\varepsilon_k \downarrow 0$ such that

$$\text{for all } N \le M \text{ in } \mathbb{N} : \ \left\|\textstyle\sum_N^M x_\iota(n)e_n - \sum_N^M x_j(n)e_n\right\| \le \varepsilon_k \quad (j \ge i \ge k) \tag{$*$}$$

Thus $x(N) = \lim_j x_j(N)$ exists for all N. Taking $j \to \infty$ in $(*)$ therefore yields:

$$\text{for all } N \le M \text{ in } \mathbb{N} : \ \left\|\textstyle\sum_N^M x_\iota(n)e_n - \sum_N^M x(n)e_n\right\| \le \varepsilon_k \quad (i \ge k)\,. \tag{$**$}$$

By $(**)$ and the fact that the series $\sum_n x_k(n)e_n$ is $\|\ \|$-Cauchy for each k, $\sum_n x(n)e_n$ is $\|\ \|$-Cauchy, so $x := \sum_1^\infty x(n)e_n$ exists. By $(**)$ $(x_\iota)_\iota$ then $\|\ \|'$-converges to x.▷

By the open graph theorem the identity map from $(E, \|\ \|)$ to $(E, \|\ \|')$ is open i.e. there is a constant $K > 0$ with $\|\ \|' \le K\ \|\ \|$, so $\|\ \|'$ is equivalent with $\|\ \|$.▷

A consequence of the continuity of basic projections is the following:

Corollary 2.5

Let $(e_n)_1^\infty$ be a basic sequence in a quasi-Banach space E.
Suppose $p_1 \le p_2 \le p_3 \le \dots$ in $\mathbb{N}$ and

$$\textstyle\sum_{j=1}^\infty \left(\sum_{n=p_j+1}^{p_{j+1}} x(n)e_n\right) =: x \quad \textit{exists.}$$

Then $x = \sum_1^\infty x(n)e_n$.

If $(e_n)_n$ is a basic sequence, then $e_n \neq 0$ for all n (by the uniqueness of expansion coefficients) and by lemma 2.4 there exists a $C \in (0,\infty)$ satisfying ($\mho$). The converse holds too:

Lemma 2.6 (Criterion Schauder basis)

Let E be a quasi-Banach space and let $(e_n)_1^\infty$ be a sequence in E.
Suppose that

(i) each vector $e_n \neq 0$;

(ii) there is a constant $K \in (0,\infty)$ such that for every choice of coefficients $(x(n))_1^\infty$ and $N \le M$ in $\mathbb{N}$:

$$\Big\|\sum_{n=1}^{N} x(n)e_n\Big\| \le K \Big\|\sum_{n=1}^{M} x(n)e_n\Big\|\,. \tag{$\bowtie$}$$

Then $(e_n)_1^\infty$ is a Schauder basis of the closure of its linear span.

◁Since both conditions are proof against passing to an equivalent quasi-norm, we may and will assume that $\|\ \|$ is r-subadditive, in particular continuous.

For a sequence of scalars $x(n)$, $n \in \mathbb{N}$, such that $\sum_1^\infty x(n)e_n$ exists, ($\bowtie$) implies that

$$\begin{aligned}\| \textstyle\sum_N^M x(n)e_n \|^r &\le \| \textstyle\sum_1^M x(n)e_n \|^r + \| -\textstyle\sum_1^N x(n)e_n \|^r \\ &\le K^r \lim_{M\to\infty} \| \textstyle\sum_{n=1}^M x(n)e_n \|^r + K^r \lim_{N\to\infty} \| \textstyle\sum_{n=1}^N x(n)e_n \|^r \le 2K^r \|x\|^r\end{aligned} \qquad (\dagger)$$

Let E_0 be the linear subspace of all $x \in E$ for which there exists a sequence of scalars $x(n)$, $n \in \mathbb{N}$, so that $x = \lim_{N\to\infty} \sum_1^N x(n)e_n$.
I: For each $x \in E_0$ there exists a unique sequence $(x(n))_n$ with $x = \sum_1^\infty x(n)e_n$.
Indeed, if $\sum_1^\infty x(n)e_n = 0$, then each $x(n) = 0$ by $(\dagger)$.
II: E_0 is closed (so the closed linear span of $\{e_n\}_n$).
Indeed, as in the proof of 2.4,

$$\|x\|' := \sup\{ \| \textstyle\sum_N^M x(n)e_n \| \ : N \le M \text{ in } \mathbb{N}\} \quad (x \in E_0)$$

defines an r-subadditive quasi-norm on E_0 such that E_0 is $\|\,\|'$-complete.
By $(\dagger)$, $\|\,\| \le \|\,\|' \le 2^{1/r}K \|\,\|$, so E_0 is $\|\,\|$-complete, and therefore $\|\,\|$-closed. ▷

Given a basic sequence we can make other basic sequences by taking 'blocks' to form new vectors with.

Definition 2.7
Let $(e_n)_{n=1}^\infty$ be a basic sequence in a quasi-Banach space E. A sequence of non-zero vectors $(u_j)_{j=1}^\infty$ is called a block basic sequence or briefly, a block basis of $(e_n)_{n=1}^\infty$ if there exist a sequence of natural numbers $0 = p_1 < p_2 < \ldots$ and of scalars $\alpha(n)$ such that

$$u_j = \textstyle\sum_{n=p_j+1}^{p_{j+1}} \alpha(n)e_n \quad (j \in \mathbb{N}).$$

By criterion 2.6, a block basis of a basic sequence is a basic sequence itself.

A quasi-Banach space can have several bases. To identify bases that are "essentially" the same, the concept of equivalence of bases is used.

Definition 2.8
Let E and F be quasi-Banach spaces. Two basic sequences, $(e_n)_1^\infty$ in E and $(f_n)_1^\infty$ in F, are called equivalent if for all sequences of scalars $(x(n))_n$:

$$\textstyle\sum_1^\infty x(n)e_n \text{ exists in } E \text{ if and only if } \sum_1^\infty x(n)f_n \text{ exists in } F\,.$$

The continuity of basic projections and the closed graph theorem imply that $(e_n)_1^\infty$ and $(f_n)_1^\infty$ are equivalent if and only if there is a linear homeomorphism $T : E \to F$ with $Te_n = f_n$. Therefore, we have

Corollary 2.9
Two basic sequences, $(e_n)_1^\infty$ in E and $(f_n)_1^\infty$ in F, are equivalent if and only if there exists a $C \in (0,\infty)$ such that for every finite sequence of scalars $(x(n))_1^N$:

$$C^{-1} \| \textstyle\sum_1^N x(n)e_n \|_E \le \| \sum_1^N x(n)f_n \|_F \le C \| \sum_1^N x(n)e_n \|_E\,.$$

The important type of basis we will be dealing with, is that of an unconditional basis. First we need to discuss the underlying concept: unconditional convergence in a quasi-Banach space (due to [RoNa]).

Proposition 2.10 (Unconditional convergence)
Let $(E, \|\|)$ be a quasi-Banach space and let $x_1, x_2, \ldots \in E$. We say that

(SI) $\sum_n x_n$ *is sign-invariant convergent if for every choice of signs* $(\theta_n \in \{-1,1\})_1^\infty$ *the series* $\sum_n \theta_n x_n$ *is convergent (to some element of E);*

(Sub) $\sum_n x_n$ *is subseries convergent if for every (strictly) increasing sequence* $n_1 < n_2 < \ldots$ *the series* $\sum_\iota x_{n_\iota}$ *is convergent (to some element of E);*

(BM) $\sum_n x_n$ *is bounded-multiplier convergent if for every bounded sequence of scalars* $(\lambda(n))_n$, *the series* $\sum_n \lambda(n) x_n$ *is convergent (to some element of E).*

Then (SI), (Sub), *and* (BM) *are equivalent, and if either of them holds, we say that* $\sum_n x_n$ *is unconditionally convergent (see also the remarks, p. 76).*

Proof
We may and will assume that $\|\|$ is r-subadditive, hence continuous.
(BM)⇒(SI) is obvious.
(SI)⇒(Sub): If $A := \{n_k\}_k$, then $\sum_1^\infty x_{n_k} = \sum_1^\infty x_n \mathbb{1}_A(n)$ and

$$\textstyle\sum_n x_n \mathbb{1}_A(n) = \frac{1}{2}[\sum_n x_n(\mathbb{1}_A - \mathbb{1}_{\mathbb{N}\setminus A})(n) + \sum_n x_n \mathbb{1}_{\mathbb{N}}(n)]\,.$$

(Sub)⇒(BM): Suppose $\sum_n x_n$ is subseries convergent. Set for $N \in \mathbb{N}$

$$\sigma_N := \sup\{\|\textstyle\sum_n x_n \mathbb{1}_A(n)\| : A \subset \mathbb{N} \text{ finite, } \min(A) \geq N\} \in [0,\infty]\,.$$

Then $\sigma_N \downarrow$ and $\sigma_1^r \leq \sum_1^N \|x_n\|^r + \sigma_{N+1}^r$ for all N.
(I) $\sigma_N \downarrow 0$, *so that (in view of the above)* $\sigma_1 < \infty$.

◁ *Suppose* $\sigma_N > \varepsilon$ *for all* $N \in \mathbb{N}$. Then there exist finite subsets $A_1, A_2, \ldots \subset \mathbb{N}$ such that $\max(A_N) < \min(A_{N+1})$ and $\|\sum_n x_n \mathbb{1}_{A_N}(n)\| \geq \varepsilon$ (all N). It follows that the subseries indicated by $A := \cup_N A_N$ is not Cauchy. *Contradiction.* ▷

(II) Let $\lambda \in \ell^\infty$, *and* $N \leq M$ *in* $\mathbb{N}$. *Then*

$$\|\textstyle\sum_{n=N}^M \lambda(n) x_n\| \leq 2\,[\sum_{k=1}^\infty 2^{-rk}]^{1/r}\,\sigma_N \,\|\,\lambda\,\|_\infty\,, \tag{F}$$

so that $\sum_n \lambda(n) x_n$ *converges and* $\|\sum_1^\infty \lambda(n) x_n\| \leq 2\,[\sum_{k=1}^\infty 2^{-rk}]^{1/r} \sigma_1 \|\,\lambda\,\|_\infty$.

◁First take $\lambda \in \ell^\infty$ with $0 \leq \lambda(n) \leq 1$. For $n \in \mathbb{N}$, write $\lambda(n)$ in the dyadic expansion as $\lambda(n) = \sum_{k=1}^\infty 2^{-k} \mathbb{1}_{\Lambda_n}(k)$. For $N, M, K \in \mathbb{N}$ with $N \leq M$:

$$\Big\|\textstyle\sum_{n=N}^M \Big(\sum_{k=1}^K 2^{-k} \mathbb{1}_{\Lambda_n}(k)\Big) x_n\Big\|^r = \Big\|\sum_{k=1}^K 2^{-k}\Big(\sum_{n=N}^M \mathbb{1}_{\Lambda_n}(k) x_n\Big)\Big\|^r$$

$$\leq \textstyle\sum_{k=1}^K (2^{-k})^r \underbrace{\Big\|\sum_{n=N}^M \mathbb{1}_{\Lambda_n}(k) x_n\Big\|^r}_{\leq \sigma_N^r} \leq [\sum_{k=1}^\infty 2^{-rk}]\,\sigma_N^r\,.$$

Taking $K \to \infty$ yields $\|\sum_N^M \lambda(n) x_n\| \leq [\sum_1^\infty 2^{-rk}]^{1/r} \sigma_N$. Since an arbitrary $\lambda \in \ell^\infty$ can be written as $\lambda = \|\,\lambda\,\|_\infty\,(\lambda^+/\|\,\lambda\,\|_\infty - \lambda^-/\|\,\lambda\,\|_\infty)$, where $\lambda^\pm(n) = \lambda(n)^\pm$, the above estimate induces (F). ▷ □

Definition 2.11
Let E be a quasi-Banach space.
A Schauder basis $(e_n)_1^\infty$ *of E is called unconditional if for each* $x \in E$, *its expansion* $\sum_1^\infty x(n) e_n$ *in terms of the basis* $(e_n)_1^\infty$ *converges unconditionally.*

Example 2.12
The standard basis $(e_n)_1^\infty$ of ℓ^p, $0 < p < \infty$, is unconditional. The summing basis in c, defined below, is not unconditional. Let

$$a_n := (\underbrace{0,0,\ldots,0}_{n-1},1,1,1,\ldots) \quad (n = 1,2,3,\ldots) .$$

Then $\| \sum_1^M x(n)a_n \|_\infty = \sup_{1\le N\le M} | \sum_{n=1}^N x(n)|$, so the conditions for being a Schauder basis are easily checked. The conditionality follows e.g. from the fact that the sequence

$$\textstyle\sum_1^N \frac{1}{n}a_n = (1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots, \sum_1^N \frac{1}{n}, 0, 0, \ldots)$$

does not converge if $N \to \infty$, while the sequence

$$\textstyle\sum_1^N \frac{1}{n}(-1)^{(n+1)}a_n = (1, 1 - \frac{1}{2}, 1 - \frac{1}{2} + \frac{1}{3}, \ldots, \sum_1^N (-1)^{n+1}\frac{1}{n}, 0, 0, \ldots)$$

does.

Proposition 2.10 yields some characterizations of unconditional bases.

Proposition 2.13 (Criteria unconditionality of a basis)
For a basic sequence $(e_n)_1^\infty$ in a quasi-Banach space, the following statements are equivalent.

(α) *$(e_n)_1^\infty$ is unconditional;*

(β) *for every choice of signs $(\theta(n))_1^\infty \in \{-1,1\}^{\mathbb{N}}$ the basic sequence $(\theta(n)e_n)_1^\infty$ is equivalent to $(e_n)_1^\infty$;*

(γ) *for every sequence of scalars λ that is bounded away from 0 and ∞ (i.e. there exist $\alpha, \beta > 0$ with $\alpha \le |\lambda(n)| \le \beta$ for all $n \in \mathbb{N}$) the basic sequence $(\lambda(n)e_n)_{n=1}^\infty$ is equivalent to $(e_n)_1^\infty$;*

(δ) *Convergence of the series $\sum_n x(n)e_n$ implies that of the series $\sum_n y(n)e_n$ whenever $|y(n)| \le |x(n)|$ for all n.*

A block basic sequence of an unconditional basic sequence is again unconditional (use e.g. the above characterization (δ) and corollary 2.5). Also, if $(f_n)_n$ and $(e_n)_n$ are equivalent bases (of F and E respectively), and one of them is unconditional, then so is the other.

We conclude with an observation that will be of eminent use in the following section: given an unconditional basis $(e_n)_1^\infty$ in E we can find an equivalent quasi-norm that is absolutely monotone with respect to the expansion coefficients.

Proposition 2.14
Let $(E, \|\!|\cdot|\!\|_0)$ be a quasi-Banach space with an unconditional basis $(e_n)_1^\infty$.
Then

$$|\!|\!|x|\!|\!| := \sup\{|\!|\!|\textstyle\sum_1^\infty \lambda(n)x(n)e_n|\!|\!|_0 : \lambda \in \ell^\infty,\ \|\lambda\|_\infty \le 1\}$$

defines an equivalent quasi-norm on E with the property

if $\sum_1^\infty x(n)e_n$ exists in E, and $(y(n))_1^\infty$ satisfies $|y(n)| \le |x(n)|$,
then $\sum_n y(n)e_n$ converges in E, and

$$|\!|\!|\textstyle\sum_1^\infty y(n)e_n|\!|\!| \le |\!|\!|\sum_1^\infty x(n)e_n|\!|\!| . \qquad \text{(absolute monotonicity)}$$

Moreover, if $|\!|\!|\cdot|\!|\!|_0$ is r-subadditive, then so is $|\!|\!|\cdot|\!|\!|$.

Proof
For $\lambda \in \ell^\infty$ the map
$$T_\lambda : x = \sum_1^\infty x(n)e_n \mapsto \sum_n \lambda(n)x(n)e_n \quad (x \in E)$$
is well-defined (since $(e_n)_n$ is unconditional), linear, and continuous (by the closed graph theorem and the continuity of the basic projections).
Further, the collection $\{T_\lambda : \ \|\lambda\|_\infty \le 1\}$ is pointwise bounded, since for fixed x the operator
$$\lambda \mapsto \sum_n \lambda(n)x(n)e_n$$
from ℓ^∞ into E is continuous (see (II) in the proof of 2.10).
Thus, by the uniform boundedness principle, the collection $\{T_\lambda : \ \|\lambda\|_\infty \le 1\}$ is uniformly bounded, and for $x \in E$
$$|\!|\!| x |\!|\!|_0 \le \sup\{|\!|\!| \textstyle\sum_1^\infty \lambda(n)x(n)e_n |\!|\!|_0 : \ \|\lambda\|_\infty \le 1\} \le (\sup_\alpha |\!|\!| T_\alpha |\!|\!|)\, |\!|\!| x |\!|\!|_0 \,.$$
The rest of the proposition is elementary. □

Remarks 2.15

Ad lemma 2.14 Zippin's proof of his characterization of perfectly homogeneous bases [Zi] uses a result of Day [Da, Theorem 1(v), p. 73] to obtain proposition 2.14 for Banach spaces. However, Day's proof is based on a convexity argument.

(Sketch of Day's argument in [Da, Theorem 1(iv), p. 73])
Given a Banach space $(E, \|\ \|_0)$ with unconditional basis $(e_n)_n$ the expression
$$\|\sum_1^\infty x(n)e_n\| =$$
$$\sup\{\|\sum_{n\in\sigma} x(n)e_n - \sum_{n\in\tau} x(n)e_n\|_0 \quad \sigma,\tau \subset \mathbb{N} \text{ finite and disjoint }\}$$
is well-defined (use the unconditionality), and since E turns out to be complete with respect to $\|\ \|$, the open mapping theorem yields that $\|\ \|$ is a norm equivalent with $\|\ \|_0$
If $\sigma,\tau \subset \mathbb{N}$ are disjoint and finite, (and $x(n)$, $n \in \sigma \cup \tau$, are scalars) then
$$t \mapsto \|\sum_{n\in\sigma} x(n)e_n + t \cdot \sum_{n\in\tau} x(n)e_n\|$$
is an even *convex* function, hence increasing in $|t|$ Thus, $|s| \le |t|$ implies that
$$\|\sum_{n\in\sigma} x(n)e_n + s \ \sum_{n\in\tau} x(n)e_n\| \le \|\sum_{n\in\sigma} x(n)e_n + t \ \sum_{n\in\tau} x(n)e_n\|$$
Using induction ($s = x(n)/y(n)$, $t = 1$) and the continuity of $\|\ \|$, we get that
$$\|\sum_1^\infty y(n)e_n\| \le \|\sum_1^\infty x(n)e_n\| \text{ if } |y(n) \le |x(n)| \text{ for all } n$$

About unconditional convergence (proposition 2.10) Let $\sum_n x_n$ be a series in a topological vector space E. We say that

(Re) $\sum_n x_n$ is reordered convergent if for every permutation π of the integers the series $\sum_n x_{\pi(n)}$ is convergent (to some element of E),

(Un) $\sum_n x_n$ is unordered convergent to x if, letting Σ be the collection of all finite subsets of $\mathbb{N}$ directed by inclusion ($\sigma \prec \sigma' \iff \sigma \subset \sigma'$), the net
$$(s_\sigma := \sum_{n\in\sigma} x_n)_{\sigma\in\Sigma}$$
converges to x i.e.
$$\forall_{\varepsilon>0}\exists_{\sigma_\varepsilon\in\Sigma} : \quad \sigma \supset \sigma_\varepsilon \Rightarrow |\!|\!| x - \sum_{n\in\sigma} x_n |\!|\!| \le \varepsilon \,.$$

In a Banach space, the notions of reordered, unordered, subseries, sign-invariant, and bounded-multiplier convergence coincide. The same is true for a quasi-Banach space, which can be proven in much the same way as for a Banach space [Si, 16.1, p. 458], except for the implication of bounded multiplier convergence by any of the other, which is done in proposition 2.10 (Sub)⇒(BM).

2.2 Zippin's theorem on perfectly homogeneous bases generalized to quasi-Banach spaces

Zippin's theorem characterizes the standard bases of ℓ^p, $p \in [1,\infty)$ and c_0 up to equivalence. Using the machinery developed in the previous section, in particular proposition 2.14, we can translate Zippin's own argument from the context of normed spaces to that of the quasi-normed spaces. The result is a characterization of the standard bases of c_0 and ℓ^p including the case that $p \in (0,1)$. The characterizing property is perfect homogeneity:

Definition 2.16
A basis $(e_n)_n$ of a quasi-Banach space is called perfectly homogeneous if it is equivalent to any of its normalized block bases.

Theorem 2.17 (Generalization of Zippin's theorem)
Let $(e_n)_n$ be a normalized basis of a quasi-Banach space $(E, \rrbracket\, \llbracket)$.
Then $(e_n)_n$ is perfectly homogeneous if and only if it is equivalent to either the standard basis of c_0 or to that of ℓ^p, for one $p \in (0,\infty)$.

Before turning to the proof of 2.17, we introduce a compact notation for "equivalence" of sequences of real numbers, which will hopefully make the reasoning more clear.

Convention 2.18
If $x = (x(n))_n$ and $y = (y(n))_n$ are two (finite or infinite) sequences of real numbers and K is a (strictly) positive real number, then (for the rest of this section) we will use the following notation

$$x \underset{K}{\sim} y \;:\Longleftrightarrow\; K^{-1}|x(n)| \le |y(n)| \le K|x(n)| \quad \textit{for all } n\,.$$

For example, if $(a_n)_n$ is a sequence in $(E, \rrbracket\, \llbracket)$, then $(\,\rrbracket a_n \llbracket\,)_n \underset{K}{\sim} (1)_n$ if and only if $(a_n)_n$ is bounded from 0 and ∞. By convention

$$x \underset{K_1}{\sim} y \underset{K_2}{\sim} z \quad \textit{means:} \quad x \underset{K_1}{\sim} y \textit{ and } y \underset{K_2}{\sim} z \textit{ (in which case } x \underset{K_1K_2}{\sim} z)\,.$$

Proof of theorem 2.17
'If': Let $p \in (0,\infty]$, and let $(e_n)_n$ be the standard basis of $c_0 \cap \ell^p$.
(i) Let $(a_n)_n$ be a block basis of $(e_n)_n$. A calculation shows that

$$\|\textstyle\sum_{n=N}^{M} x(n)a_n\|_p = \|\sum_{n=N}^{M} x(n)\, \|a_n\|_p\, e_n\|_p \quad (x(n)\,,\ n \in \mathbb{N}, \text{ scalars}) \qquad (\#)$$

Therefore, if $(\,\|a_n\|_p\,)_n \underset{K}{\sim} (1)_n$, then $(a_n)_n$ is equivalent to $(e_n)_n$.
(ii) Suppose that $(\hat{e}_n)_n$ is a basis of $(E, \rrbracket\, \llbracket)$ that is equivalent to $(e_n)_n$. Then there is a linear homeomorphism $T : (E, \rrbracket\, \llbracket) \to (c_0 \cap \ell^p, \|\;\|_p)$ with $T(\hat{e}_n) = e_n$ (all n), and $\|T(\hat{e})\|_p \underset{K}{\sim} \rrbracket \hat{e} \llbracket$ ($\hat{e} \in E$) (see 2.9).
Let $(\hat{a}_n)_n$ be a $\rrbracket\, \llbracket$-normalized block basis of $(\hat{e}_n)_n$. We prove that it is equivalent to $(\hat{e}_n)_n$. Observe that the $T(\hat{a}_n) =: a_n$, $n \in \mathbb{N}$, form a block basis of $(e_n)_n$ and that $(\,\|a_n\|_p\,)_n \underset{K}{\sim} (\rrbracket \hat{a}_n \llbracket)_n = (1)_n$. Then subsequently: $(\hat{a}_n)_n$ is equivalent with $(a_n)_n$ (via T); $(a_n)_n$ is equivalent with $(e_n)_n$ (via (#), since $(\,\|a_n\|_p\,)_n \underset{K}{\sim} (1)_n$); $(e_n)_n$ is equivalent with $(\hat{e}_n)_n$ (by assumption).

'Only-if': Suppose that $(e_n)_n$ is $\|\!\|\ \|\!\|$-normalized and perfectly homogeneous.
First observe that $(e_n)_n$ is an unconditional basis (2.13 (β)). Therefore, if $\|\!\|\ \|\!\|_0$ is equivalent to $\|\!\|\ \|\!\|$, then $(e_n)_n$ in $(E, \|\!\|\ \|\!\|)$ is equivalent with $(e_n / \|\!\|e_n\|\!\|_0)_n$ in $(E, \|\!\|\ \|\!\|_0)$ (use 2.13 (γ) and the fact that $\|\!\|\ \|\!\|$ is equivalent to $\|\!\|\ \|\!\|_0$). We will use that and proposition 2.14 to assume that $\|\!\|\ \|\!\|$ is r-subadditive and absolutely monotone in the expansion coefficients (with respect to $(e_n)_n$).

Next, we strengthen the equivalence of $(e_n)_n$ with each of its normalized block bases to a uniform equivalence of $(e_n)_n$ with all of its block bases i.e. there is a constant $M > 0$ such that

$$M^{-1} \|\!\|\textstyle\sum_{\iota=1}^{n} x(\iota)u_\iota\|\!\| \le \|\!\|\sum_{\iota=1}^{n} x(\iota)e_\iota\|\!\| \le M \|\!\|\sum_{\iota=1}^{n} x(\iota)u_\iota\|\!\|$$

for all $n \in \mathbb{N}$, all scalars $(x(\iota))_\iota$, and all normalized block bases $(u_\iota)_\iota$,

the proof of which will be postponed to lemma 2.19 on page 79.

Lastly, we need to come up with a $p \in (0, \infty)$ for which $(e_n)_n$ is equivalent to the standard basis of $l^p \cap c_0$. To this end, we set

$$s(n) := \|\!\|\textstyle\sum_1^n e_\iota\|\!\| \qquad (n \in \mathbb{N}),$$

because we expect that $s(n) \sim n^{1/p}$ (with the convention that $1/p = 0$ if $p = \infty$). Using the uniform equivalence of $(e_n)_n$ with its block bases, applied to well chosen block-bases and ditto coefficients, we will deduce in lemma 2.20 (p. 81) further on that

$$M^{-2k}\, s(n)^k \le s(n^k) \le M^{2k}\, s(n)^k \qquad (k, n \in \mathbb{N}).$$

which, as we will see ibidem, implies that there exists a $p \in (0, \infty]$ such that

$$M^{-2} n^{1/p} \le s(n) \le M^2 n^{1/p} \qquad (n \in \mathbb{N}). \tag{$\bowtie$}$$

To proceed, we distinguish two cases: $p = \infty$ and $p \in (0, \infty)$.

Case $p = \infty$.
If $p = \infty$, we have $\|\!\|\sum_1^n e_\iota\|\!\| \le M^2$ (all n). Let $y \in c_{00}$, say $y = \sum_1^N y(n)e_n$. Using the absolute monotonicity of $\|\!\|\ \|\!\|$ the following chain of inequalities uncoils

$$\begin{aligned}\|\!\|y\|\!\| &= \Big\|\!\Big\|\textstyle\sum_1^N y(\iota)e_\iota\Big\|\!\Big\| \le \Big\|\!\Big\|\sum_1^N \|y\|_\infty e_\iota\Big\|\!\Big\| = \|y\|_\infty \Big\|\!\Big\|\sum_1^N e_\iota\Big\|\!\Big\| \\ &\le M^2 \|y\|_\infty .\end{aligned}$$

So, if $x \in c_0$, then $\sum_n x(n)e_n$ is $\|\!\|\ \|\!\|$-Cauchy. On the other hand, if $\sum_1^\infty x(n)e_n$ exists, $|x(n)| = \|\!\|x(n)e_n\|\!\|$ converges to zero. Therefore, $\sum_1^\infty x(\iota)e_\iota$ exists if and only if $x \in c_0$, which establishes the equivalence of $(e_n)_n$ with the standard basis of c_0.

Before handling the case $p \in (0, \infty)$, we reformulate the relevant results using the $\sim$-notation.

i. The uniform equivalence of all normalized block bases (lemma 2.19) yields:
 if $(u_n)_n$ is a normalized block basis of $(e_n)_n$, then

$$\|\!\|\textstyle\sum_1^n x(\iota)e_\iota\|\!\| \underset{M}{\sim} \|\!\|\sum_1^n x(\iota)u_\iota\|\!\|$$

 for all $n \in \mathbb{N}$, and all sequences of scalars $(x(n))_n$;

ii. the absolute monotonicity of $\|\!|\ |\!\|$ yields

$$(x(i))_\iota \underset{K}{\sim} (y(i))_\iota \quad \Rightarrow \quad \|\textstyle\sum_1^n x(i)e_\iota\| \underset{K}{\sim} \|\sum_1^n y(i)e_\iota\| \ ;$$

iii. equation ($\bowtie$) (to be proven in lemma 2.20 page 81) reads:

$$\Big\| \textstyle\sum_1^n e_\iota \Big\| \underset{M^2}{\sim} n^{1/p} \quad \text{for all } n \in \mathbb{N}.$$

We now return to the remaining case.

Case $0 < p < \infty$.
Take $n, k_1, k_2, \dots k_n \in \mathbb{N}$. Then

$$\begin{aligned}(k_1^{1/p}, k_2^{1/p}, \dots, k_n^{1/p}) &\underset{M^2}{\overset{\text{iii}}{\sim}} (\,\|\textstyle\sum_1^{k_1} e_\iota\|, \|\sum_1^{k_2} e_\iota\|, \dots, \|\sum_1^{k_n} e_\iota\|\,) \\ &\underset{M}{\overset{\text{i,ii}}{\sim}} (\,\|\textstyle\sum_1^{k_1} e_\iota\|, \|\sum_{k_1+1}^{k_2} e_\iota\|, \dots, \|\sum_{k_1+\dots+k_{n-1}+1}^{k_1+\dots+k_n} e_\iota\|\,).\end{aligned}$$

Therefore, (with $k_0 := 1$)

$$\begin{aligned}\Big\|\textstyle\sum_1^n k_\iota^{1/p} e_\iota\Big\| &\underset{M^3}{\overset{\text{ii}}{\sim}} \Big\|\textstyle\sum_{\iota=1}^n \Big(\|\sum_{j=k_0+\dots+k_{\iota-1}}^{k_0+\dots+k_\iota} e_j\|\Big) e_\iota\Big\| \\ &\underset{M}{\overset{\text{i}}{\sim}} \Big\|\textstyle\sum_{\iota=1}^n \Big(\|\sum_{j=k_0+\dots+k_{\iota-1}}^{k_0+\dots+k_\iota} e_j\|\Big) \dfrac{\sum_{j=k_0+\dots+k_{\iota-1}}^{k_0+\dots+k_\iota} e_j}{\|\sum_{j=k_0+\dots+k_{\iota-1}}^{k_0+\dots+k_\iota} e_j\|}\Big\| \\ &= \Big\|\textstyle\sum_1^{k_1+\dots+k_n} e_\iota\Big\| \underset{M^2}{\overset{\text{iii}}{\sim}} [k_1 + \dots + k_n]^{1/p}.\end{aligned}$$

Scraping together all factors M, we conclude

$$\Big\|\textstyle\sum_1^n x(i)e_\iota\Big\| \underset{M^6}{\sim} [x(1)^p + \dots + x(n)^p]^{1/p} \quad (n \in \mathbb{N}) \tag{†}$$

if $x(i) = k_\iota^{1/p}$ with $k_\iota \in \mathbb{N}$. Since $\|\!|\ |\!\|$ is absolutely homogeneous, (†) also holds if $x(i) = (k_\iota/m)^{1/p}$ with $k_\iota, m \in \mathbb{N}$, and, by the continuity of $x \mapsto \|x\|$, (†) in fact holds for $x(i) \in \mathbb{R}^+$. Finally, because $\|\!|\ |\!\|$ is absolutely monotone,

$$\|\textstyle\sum_1^n x(i)e_\iota\| = \|\sum_1^n |x(i)|e_\iota\| \underset{M^6}{\sim} [|x(1)|^p + \dots + |x(n)|^p]^{1/p}$$

for all $n \in \mathbb{N}$, and all sequences of scalars $(x(i))_\iota$.
Thus, $\sum_n x(n)e_n$ exists if and only if $x \in \ell^p$, which establishes the equivalence of $(e_n)_n$ with the standard basis of ℓ^p. □

We conclude with the proofs of lemmas 2.19 and 2.20.

Lemma 2.19 (Uniform equivalence of normalized block basic sequences)
Let $(E, \|\!|\ |\!\|)$ be a quasi-Banach space with basis $(e_n)_n$.
Suppose that $(e_n)_n$ is perfectly homogeneous and that $\|\!|\ |\!\|$ is r-subadditive (for some $r \in (0,1]$) and absolutely monotone (with respect to $(e_n)_n$).
If $\mathsf{u} = (u_n)_n$ is a normalized block basis of $(e_n)_n$, let $T_\mathsf{u} : E \to E$ defined by

$$x = \textstyle\sum_1^\infty x(n)e_n \mapsto \sum_1^\infty x(n)u_n \quad (x \in E)$$

be the linear homeomorphism that exhibits the equivalence of $(e_n)_n$ with $(u_n)_n$. Then

$$M := \sup\left\{ \|T_\mathsf{u}\|, \|T_\mathsf{u}^{-1}\| : \begin{array}{l}\mathsf{u} = (u_n)_n \text{ is a normalized} \\ \text{block basis of } (e_n)_n\end{array} \right\} < \infty .$$

Proof
First observe that for a(ny) normalized block basis $(u_j)_j$ of $(e_n)_n$:

i. $u_j \in [[e_j, e_{j+1}, \ldots]]$ for all $j \in \mathbb{N}$ (use induction);

ii. $(u_j)_j$ is equivalent to any other normalized block basis $(v_j)_j$ of $(e_n)_n$.

(I: $\sup\{ \|T_{\mathrm{u}}^{-1}\| : \mathrm{u} = (u_j)_j$ is a normalized block basis of $(e_n)\} < \infty$).
Suppose not. Then

$$\sup\left\{ \left\|\sum_1^\infty x(n)e_n\right\| : \begin{array}{l}(x(n))_n \text{ is a sequence of scalars such} \\ \text{that there is a normalized block basis} \\ (u_n)_n \text{ with } \|\sum_1^\infty x(n)u_n\| \le 1\end{array}\right\} = \infty . \qquad (\#)$$

From that, we will show below that there exist a sequence of vectors $u_{p_1}, \ldots, u_{q_1}$, $u_{p_2}, \ldots, u_{q_2}, u_{p_3}, \ldots, u_{q_3}, \ldots$ and a sequence of scalars $x(p_1), \ldots, x(q_1), x(p_2)$, $\ldots, x(q_2), x(p_3), \ldots, x(q_3), \ldots$ satisfying

$$\left.\begin{array}{l} 1 =: p_1 \le q_1 < p_2 \le \ldots, \text{ and for all } j \in \mathbb{N} : \\ (u_n)_{n=p_j}^{q_j} \text{ is a finite normalized block basis of } (e_n)_{n=p_j}^{p_{j+1}-1} , \\ \left\|\sum_{n=p_j}^{q_j} x(n)e_n\right\| \ge 1 , \text{ while } \left\|\sum_{n=p_j}^{q_j} x(n)u_n\right\| \le 2^{-j} . \end{array}\right\} \qquad (\ddagger)$$

In terms of the official definition 2.7, the second line of $(\ddagger)$ is to be understood as $(v_i := u_{i+p_j-1})_{i=1}^{q_j-p_j+1}$ is a finite block basic sequence of $(w_k := e_{k+p_j-1})_{k=1}^{p_{j+1}-p_j})$.

Then $u_{p_1}, \ldots, u_{q_1}, u_{p_2}, \ldots$, is a normalized block basis, and as such equivalent to $e_{p_1}, \ldots, e_{q_1}, e_{p_2}, \ldots$. Thus there exists a $K > 0$ such that

$$1 \le \left\|\sum_{n=p_j}^{q_j} x(n)e_n\right\| \le K \left\|\sum_{n=p_j}^{q_j} x(n)u_n\right\| \quad (\text{for all } j \in \mathbb{N}) ,$$

which yields a *contradiction.*

The construction of $(u_n)_{p_j}^{q_j}$ and $(x(n))_{p_j}^{q_j}$ proceeds via induction. For the induction step first observe that (#) can be strengthened to: for all $N \in \mathbb{N}$

$$\sup\left\{ \left\|\sum_N^\infty x(n)e_n\right\| : \begin{array}{l}x \text{ a sequence of scalars such that} \\ \text{there exists a normalized block basis} \\ (u_n)_n \text{ with } \|\sum_1^\infty x(n)u_n\| \le 1\end{array}\right\} = \infty . \qquad (!\#)$$

$\triangleleft$ Indeed, if there were an N for which the supremum in (!#) was finite (say less than K), then for any sequence of scalars $x(n))_n$ and normalized block basis $(u_n)_n$ with $\|\sum_1^\infty x(n)u_n\| = 1$ we have $|x(n)| = \|x(n)u_n\| \le \|\sum_1^\infty x(n)u_n\| = 1$ so that

$$\|\sum_1^\infty x(n)e_n\|^r \le \sum_1^{N-1} |x(n)|^r + \|\sum_N^\infty x(n)e_n\|^r \le (N-1) + K^r .$$

$\triangleright$

As for the induction step: suppose p_{i+1}, q_i, $(u_n)_{n=p_i}^{q_i}$, $(x_n)_{n=p_i}^{q_i}$ have been constructed for all $i \le j-1$ in $\mathbb{N}$. Using (!#) we select a normalized block basis $(u_n)_n$ and a sequence of scalars $(x(n))_n$ with

$$\|\sum_1^\infty x(n)u_n\| = 2^{-j} \quad \text{and} \quad \|\sum_{p_j}^\infty x(n)e_n\| > 1 .$$

Using the continuity of $\|\|$, we choose $q_j \ge p_j$ so large that

$$\|\sum_{p_j}^{q_j} x(n)e_n\| \ge 1 .$$

By the absolute monotonicity of $\|\|$, we retain

$$\left\| \sum_{p_1}^{q_1} x(n)u_n \right\| \le \left\| \sum_1^\infty x(n)u_n \right\| = 2^{-1} .$$

Finally, we choose a $p_{j+1} > q_j$ such that $(u_n)_{n=p_j}^{q_j}$ is a finite block basic sequence of $(e_n)_{n=p_j}^{p_{j+1}-1}$.

(II: $\sup\{ \|T_u\| : u = (u_j)_j$ *normalized block basis of* $(e_n)\} < \infty$ *.)*
Suppose not. With arguments similar to that in (I), we obtain that for all $N \in \mathbb{N}$

$$\sup\left\{ \left\| \sum_N^\infty x(n)u_n \right\| : \begin{array}{l} (x(n))_n \text{ is a sequence of scalars such} \\ \text{that } \|\sum_1^\infty x(n)e_n\| \le 1 \text{ , and} \\ (u_n)_n \text{ is a normalized block basis} \end{array} \right\} = \infty ,$$

and that can be used to define inductively a sequence of vectors $u_{p_1}, \dots, u_{q_1}, u_{p_2}, \dots, u_{q_2}, u_{p_3}, \dots$ and a sequence of scalars $x(p_1), \dots, x(q_1), x(p_2), \dots, x(q_2), x(p_3), \dots$ such that

$$\left. \begin{array}{l} 1 =: p_1 \le q_1 < p_2 \le \dots, \text{ and for all } j \in \mathbb{N} : \\ (u_n)_{n=p_j}^{q_j} \text{ is a finite normalized block basis of } (e_n)_{n=p_j}^{p_{j+1}-1} , \\ \left\| \sum_{n=p_j}^{q_j} x(n)u_n \right\| \ge 1 \text{ , while } \left\| \sum_{n=p_j}^{q_j} x(n)e_n \right\| \le 2^{-j} , \end{array} \right\}$$

which *contradicts* the equivalence of $u_{p_1}, \dots, u_{q_1}, u_{p_2}, \dots$ and $e_{p_1}, \dots, e_{q_1}, e_{p_2}, \dots$

□

One lemma to go.

Lemma 2.20 (The growth of $\|e_1 + \dots + e_n\|$ is proportional to $n^{1/p}$)
Let $(E, \|\|)$ be quasi-Banach space with normalized basis $(e_n)_n$.
Suppose that $(e_n)_n$ is perfectly homogeneous and that $\|\|$ is absolutely monotone and r-subadditive for some $r \in (0, 1]$.
Set

$$s(n) := \left\| \sum_1^n e_i \right\| \quad (n \in \mathbb{N}).$$

Then the sequence $s(1), s(2), \dots$ is increasing and obeys the relation

$$M^{-2}s(n)s(m) \le s(nm) \le M^2 s(n)s(m) \quad \textit{for all } m, n \in \mathbb{N} ,$$

with M as in lemma 2.19. As a result

i. $\lim_{n\to\infty} \dfrac{s(n)}{\log n} =: 1/p \in [0, \infty)$ *exists ($1/p = 0$ corresponds to $p = \infty$), and*

ii. $M^{-2} n^{1/p} \le s(n) \le M^2 n^{1/p} \quad (n \in \mathbb{N})$. $\qquad (\bowtie)$

Proof
For notational convenience, we use the notation $\underset{K}{\sim}$ (see page 77) again.
That $(s(n))_n$ is increasing, follows from the absolute monotonicity of $\|\|$.
Moreover, of the results mentioned on page 78, we have i. (the uniform equivalence of $(e_n)_n$ with all of its block basic sequences) and ii. (the absolute monotonicity of $\|\|$) at our disposal. We will refer to them with i. and ii. accordingly.

Let $n, m \in \mathbb{N}$. By the uniform equivalence of normalized block basic sequences,

$$\Big(\|\sum_1^m e_\iota\| \,,\; \|\sum_{m+1}^{2m} e_\iota\| \,,\; \ldots \,,\; \|\sum_{(n-1)m+1}^{nm} e_\iota\| \Big) \underset{M}{\sim} \Big(s(m), s(m), \ldots, s(m) \Big) \,, \qquad (\%)$$

so that

$$s(nm) = \Big\|\sum_1^{nm} e_\iota\Big\| = \Big\|\sum_1^m e_\iota + \sum_{m+1}^{2m} e_\iota + \ldots + \sum_{(n-1)m+1}^{nm} e_\iota\Big\|$$

$$= \Big\| \|\textstyle\sum_1^m e_\iota\| \cdot \dfrac{\sum_1^m e_\iota}{\|\sum_1^m e_\iota\|} + \|\sum_{m+1}^{2m} e_\iota\| \cdot \dfrac{\sum_{m+1}^{2m} e_\iota}{\|\sum_{m+1}^{2m} e_\iota\|} + \ldots$$

$$\ldots + \|\textstyle\sum_{(n-1)m+1}^{nm} e_\iota\| \cdot \dfrac{\sum_{(n-1)m+1}^{nm} e_\iota}{\|\sum_{(n-1)m+1}^{nm} e_\iota\|} \Big\|$$

$$\overset{\text{I.}}{\underset{M}{\sim}} \Big\| \|\textstyle\sum_1^m e_\iota\|\, e_1 + \|\sum_{m+1}^{2m} e_\iota\|\, e_2 + \ldots + \|\sum_{(n-1)m+1}^{nm} e_\iota\|\, e_n \Big\|$$

$$\overset{(\%)\,\text{II.}}{\underset{M}{\sim}} \Big\|s(m)e_1 + s(m)e_2 + \ldots + s(m)e_n\Big\| = s(m)\,\Big\|e_1 + \ldots + e_n\Big\|$$

$$= s(m)s(n) \qquad \text{i.e.} \quad M^{-2}s(n)s(m) \le s(nm) \le M^2 s(n)s(m) \,.$$

Induction yields:

$$M^{-2k}s(n)^k \le s(n^k) \le M^{2k}s(n)^k \quad (n, k \in \mathbb{N}) \,. \qquad (\&)$$

To calculate $\lim_n \log s(n)/\log n$, fix $n, m \in \mathbb{N}$.
For $k \in \mathbb{N}$ let $l = l(k)$ be the entier of $k \log n/\log m$ i.e. $m^l \le n^k \le m^{l+1}$.
Then

$$\lim_{k\to\infty} \frac{l(k)}{k} = \frac{\log n}{\log m} \,, \qquad (\natural)$$

and, since the sequence $(s(n))_n$ is increasing, $s(m^l) \le s(n^k)$.
The latter implies, together with (&),

$$M^{-2l}\, s(m)^l \le M^{2k}\, s(n)^k \,.$$

Taking the logarithm on both sides of the inequality above, dividing by k and taking the limit $k \to \infty$ using ($\natural$), yields

$$\frac{\log n}{\log m}\Big(-2\log M + \log s(m)\Big) \le 2\log M + \log s(n) \,.$$

Conclusion:

$$\frac{\log s(m)}{\log m} - \frac{\log s(n)}{\log n} \le 2\log M\Big(\frac{1}{\log n} + \frac{1}{\log m}\Big) \qquad (\text{for all } n, m \in \mathbb{N}) \,,$$

which yields, by the symmetric rôles of n and m:

$$\left| \frac{\log s(m)}{\log m} - \frac{\log s(n)}{\log n} \right| \le 2\log M \left(\frac{1}{\log n} + \frac{1}{\log m} \right) \qquad (n, m \in \mathbb{N}) \qquad (\flat)$$

Thus, the sequence $(\log s(m)/\log m)_m$ is Cauchy, say

$$1/p = \lim_{m\to\infty} \frac{\log s(m)}{\log m} \in [0,\infty) .$$

Letting $m \to \infty$ in ($\flat$) and exponentiating the result exhibits

$$M^{-2} n^{1/p} \le s(n) \le M^2 n^{1/p} \quad (n \in \mathbb{N}) ,$$

which concludes the proof of the second step. □

Remarks 2.21

Ad theorem 2.17 The proof of lemma 2.20 only uses normalized block bases with constant coefficients (i.e. $\alpha(n) = \alpha(m)$ if $p_j < n, m \le p_{j+1}$), while the proof of 2.19 works equally well if we assume that $(e_n)_n$ is equivalent to any of its normalized block bases with constant coefficients. Conclusion: we can weaken the assumption of perfect homogeneity in theorem 2.17 to the assumption that $(e_n)_n$ is equivalent to any of its normalized block bases with constant coefficients.

2.3 Quasi-normed Riesz spaces

In the next two sections we will generalize an isomorphic characterization of L^p-spaces and c_0-spaces ([LiTz2, theorem 1.b.13, p. 22]).

For the characterization we will take into account

1. the metric structure of these spaces, which is that of a quasi-normed space,
2. the ordering structure, which is that of a Riesz space, and
3. the close relation between those two structures, which ensues from the way the generator of the metric structure, $⦀\ ⦀ := \|\ \|_p$, pays respect to the ordering:

$$|x| \le |y| \quad \Longrightarrow \quad ⦀x⦀ \le ⦀y⦀ \quad (x, y \in E) . \tag{Riesz}$$

We will end up with characterizing L^p-spaces and M-spaces among the class of quasi-normed Riesz spaces.

Definition 2.22

A Riesz quasi-norm is a quasi-norm on a Riesz space that is Riesz as above.
A quasi-normed Riesz space is a Riesz space equipped with a Riesz quasi-norm.
A quasi-Banach lattice is a quasi-normed Riesz space that is complete with respect to the topology generated by its Riesz quasi-norm.

The characterization we are aiming at, will be an characterization up to a topological Riesz isomorphism:

Definition 2.23

Two quasi-normed Riesz spaces $(E, ⦀\ ⦀_E)$ and $(F, ⦀\ ⦀_F)$ are called topologically Riesz isomorphic if there exists a Riesz isomorphism Ω from E onto F that is at the same time a homeomorphism: there exists a $K \in (0,\infty)$ such that

$$K^{-1}\, ⦀x⦀_E \le ⦀\Omega(x)⦀_F \le K\, ⦀x⦀_E \quad \textit{for all } x \in E .$$

For use further on, we collect some properties of quasi-normed Riesz spaces.

First of all, since a quasi-normed Riesz space is endowed with the structure of a quasi-normed space, the notion of a basic sequence makes sense. The available ordering structure allows to distinguish a special type of it.

Lemma 2.24
Let E be a quasi-normed Riesz space.

i. *Every disjoint sequence is a basic sequence;*

ii. *If E is infinite dimensional, it contains an infinite disjoint sequence of non-zero vectors, and hence an infinite basic sequence.*

$\lhd$ (i) follows from lemma 2.6, the Riesz property and the fact that for disjoint $e_1, e_2, \ldots$:
$$|\sum_1^N \lambda(i)e_i| = \sum_1^N |\lambda(i)e_i| \le \sum_1^M |\lambda(i)e_i| = |\sum_1^M \lambda(i)e_i| \quad (N \le M,\ \lambda(i) \text{ scalars}).$$
(ii) is lemma 0.42. $\rhd$

Topological completeness of a quasi-normed Riesz space adds some interesting features:

Lemma 2.25
Let $(E, \|\ \|)$ be a quasi-Banach lattice. Then

i. *Closed Riesz subspace and ideals of E are uniformly complete. In particular, every principal ideal is Riesz isomorphic to a $C(X)$ for some compact Hausdorff space X;*

ii. *if $Q \subset E$ is countable, then there is a principal ideal of E that contains Q.*
In particular, if $\|\ \|_1$ and $\|\ \|_2$ are two Riesz quasi-norms on E, then:
$\|\ \|_1$ is equivalent to $\|\ \|_2$ if and only if for every principal ideal D of E the restrictions of $\|\ \|_1$ and $\|\ \|_2$ to D are equivalent.

$\lhd$ (i) Let E_0 be a Riesz subspace. We check when the uniform completeness criterion
$$0 \le u_n \le 2^{-n}u \text{ in } E_0^+ \quad \Longrightarrow \quad E_0\text{-}\sup_N \sum_1^N u_n \text{ exists in } E_0,$$
(p. 19) is satisfied. Well, such $(u_n)_n$ have $\|u_n\| \le 2^{-n}\|u\|$, so $\|\ \|\text{-}\sum_1^\infty u_n$ exists in E (by 0.99), and is automatically the supremum of $\sum_1^N u_n$, $N \in \mathbb{N}$, according to lemma 1.4. Further, either closedness or solidness of E_0 ($\sum_1^N u_n \le 2u$, all N) imply that $\sup_N \sum_1^N u_n = \|\ \|\text{-}\lim_N \sum_1^N u_n$ lies in E_0.
(ii) If $Q = \{q_n\}_n$, then the ideal generated by $e := \sum_1^\infty 2^{-n} q_n \|q_n\|^{-1}$ contains Q.
$\rhd$

A quasi-normed Riesz space E has two duals: one as quasi-normed space, denoted by E', and one as Riesz space, denoted by $E^\sim$. They are closely related:

Lemma 2.26
Let $(E, \|\ \|)$ be a quasi-normed Riesz space.
Then the topological dual E' is an ideal of the order dual $E^\sim$, and $\|\ \|'$ is a Riesz norm on E'. Thus, $(E', \|\ \|')$ is a Banach lattice.
If E is in addition a quasi-Banach lattice, then $E' = E^\sim$.

$\lhd$ Let $\phi \in E'$, and $u \in E^+$. Then for $x \in [-u, u]$: $|\phi(x)| \le \|\phi\|' \|x\| \le \|\phi\|' \|u\|$, so $\phi[-u, u]$ is order bounded in $\mathbb{R}$, whence $\phi \in E^\sim$.
Further, if $\psi \in E^\sim$ with $|\psi| \le |\phi|$ then $|\psi(x)| \le |\psi|(|x|) \le |\phi|(|x|) \le \|\phi\|\ \|x\|$ (0.84), so $\psi \in E'$ (E' is an ideal of $E^\sim$) and $\|\psi\|' \le \|\phi\|'$ ($\|\ \|'$ is a Riesz norm).
Finally, assume that E is $\|\ \|$-complete.
If there exists a $\phi \in (E^\sim)^+ \setminus (E')^+$, then there are $x_1, x_2, \ldots \in E$ such that $\|x_n\| \le 2^{-n}$, while $|\phi(x_n)| \ge 2^n$. The sum $u := \sum |x_n|$ exists by topological completeness of E, but $\phi(u) \ge \phi(|x_n|) = |\phi|(|x_n|) \ge |\phi(x_n)| \ge 2^n$ all n. *Contradiction.* $\rhd$

2.4 Characterizations of L^p-spaces and quasi-M-spaces

Specific for an L^p-space (M-space) is that every disjoint normalized sequence is equivalent to the standard basis of ℓ^p (c_0), which is a consequence of the p-additivity.

In this section and the next one we will study the converse question: is a quasi-Banach lattice in which disjoint normalized sequences are equivalent (topologically Riesz) isomorphic to an L^p-space or an M-space?

For finite dimensional Riesz spaces the answer is degenerately affirmative.

Lemma 2.27

Let E be a Riesz space of dimension $d < \infty$, and let τ be a vector space topology on E. Then (E, τ) is topologically Riesz isomorphic to $(\ell^p(1, \dots, d), \| \ \|_p)$ for all $p \in (0, \infty]$.

The proof of the lemma above essentially consists of two observations.

1. If E is a finite dimensional Riesz space of dimension d, then there exists a Riesz isomorphism ϕ from $(\mathbb{R}^d, \leq)$ onto $(E, \leq)$ (0.32). In particular ϕ is a linear isomorphism.
2. If E is a finite dimensional topological vector space of dimension d, then any linear isomorphism from $\mathbb{R}^d$ onto E is in fact a linear homeomorphism from $(\mathbb{R}^d, \| \ \|_2)$ onto (E, τ) (see e.g. [Ed, 1.9.6, p. 64]).

In view of the above lemma we will confine ourselves for the rest of this section to *infinite dimensional* quasi-Banach lattices.

Since disjoint normalized sequences will appear more than frequently in what follows, we reserve a special name for them.

Definition 2.28

Let $(E, \|\!\|\ \|\!\|)$ be a quasi-normed Riesz space. A (finite or infinite) sequence $(u_i)_i$ in E^+ is called orthonormal if $u_i \wedge u_j = 0$ for $i \neq j$ and $\|\!\| u_i \|\!\| = 1$ for all i.

A crucial rôle in this section is played by the following lemma.

Lemma 2.29

Let E be an infinite dimensional quasi-Banach lattice such that every two infinite orthonormal sequences are equivalent. Then the finite orthonormal sequences are uniformly equivalent, i.e.

$$\left.\begin{array}{c}\text{there is a constant } M \in [1, \infty) \text{ such that} \\ M^{-1} \|\!\| \sum_1^n \lambda(i) v_i \|\!\| \leq \|\!\| \sum_1^n \lambda(i) u_i \|\!\| \leq M \|\!\| \sum_1^n \lambda(i) v_i \|\!\| \\ \text{for all finite orthonormal sequences } (u_i)_{i=1}^n \text{ and } (v_i)_{i=1}^n \text{ in } E, \\ \text{and all sequences of scalars } (\lambda(i))_{i=1}^n .\end{array}\right\} \qquad (\pitchfork)$$

In particular, all orthonormal infinite sequences are uniformly equivalent.

We postpone the proof of lemma 2.29 to the end of this section, and first discuss its consequences for characterizing L^p-spaces and for, what we will call, quasi-M-spaces.

Let E be an infinite dimensional quasi-Banach lattice such that ($\pitchfork$) of lemma 2.29 above holds. Take an infinite orthonormal sequence $(e_i)_1^\infty$ in E (see lemma 2.24).

By $(\pitchfork)$, $(e_n)_n$ is perfectly homogeneous, and so there are by Zippin's theorem (p. 77) a $p \in (0, \infty]$ and an $M_1 \in (0, \infty)$ such that for all finite sequences of scalars $(\lambda(i))_1^n$

$$M_1{}^{-1}\ \|(\lambda(i))_1^n\|_p \leq \Big\|\sum_1^n \lambda(i) e_i\Big\| \leq M_1\ \|(\lambda(i))_1^n\|_p\ .$$

Combining the above inequalities with $(\pitchfork)$, results in

Lemma 2.30

Let E be an infinite dimensional quasi-Banach lattice in which the finite orthonormal sequences are uniformly equivalent (i.e. $(\pitchfork)$ holds).
Then there is a $p \in (0, \infty]$ and a constant $K \in [1, \infty)$ such that

$$K^{-1} \Big\|\sum_1^n x_\iota\Big\| \leq \Big\|(\ \|x_\iota\|\ )_1^n\Big\|_p \leq K \Big\|\sum_1^n x_\iota\Big\|$$

for each disjoint finite sequence $(x_\iota)_1^n$ in E.

We will now examine the cases $p < \infty$ and $p = \infty$ (of the above lemma) separately. As we will see there is a drastic difference between those two cases. A first taste of this difference is the following:

if in the setting of above $p < \infty$, then E is conditionally σ-laterally complete and hence possesses the principal projection property (0.64).

$\lhd$ Indeed, let $(u_n)_1^\infty$ be a disjoint sequence in E^+ majorized by $v \in E^+$. Using lemma 2.30 we see that for all $N \leq M$ in $\mathbb{N}$:

$$\|v\| \geq \Big\|\sum_N^M u_n\Big\| \underset{K}{\sim} \Big\|(\ \|u_n\|\ )_N^M\Big\|_p = [\textstyle\sum_N^M \|u_n\|^{\,p}]^{1/p}\ .$$

From this we infer successively that $[\sum_1^\infty \|u_n\|^p]^{1/p} < \infty$ and that the series $\sum_n u_n$ is $\|\ \|$-Cauchy. Its quasi-norm limit $\sum_1^\infty u_n$ is the supremum of $\{u_n : n \in \mathbb{N}\}$ (by lemma 1.4 on page 47). $\rhd$

In contrast, observe that the case $p = \infty$ includes M-spaces which are not (conditionally) σ-laterally complete (such as $C[0,1]$).

The case $p < \infty$ allows a satisfactory characterization.

Theorem 2.31 (Characterization of infinite dimensional L^p-spaces)

Let $p \in (0, \infty)$ and let $(E, \|\ \|)$ be an infinite dimensional quasi-Banach lattice such that every infinite orthonormal sequence is equivalent to the standard basis of ℓ^p. Write "$\sum_1^n x_\iota = x$ disjoint" if $\sum_1^n x_\iota = x$ and $(x_\iota)_\iota$ is a disjoint sequence.
Then

i. the formula

$$\|x\|_\triangledown := \inf\Big\{ [\textstyle\sum_1^n \|x_\iota\|^p]^{1/p}\ :\ n \in \mathbb{N},\ \sum_1^n x_\iota = x \text{ disjoint}\Big\} \quad (x \in E)\ ,$$

introduces a Riesz quasi-norm that is equivalent to $\|\ \|$ and that is p-subadditive on disjoint vectors i.e.

$$\|x + y\|_\triangledown^p \leq \|x\|_\triangledown^p + \|y\|_\triangledown^p \quad (x, y \in E, x \wedge y = 0)\ .$$

Moreover, if $\|\ \|$ was p-superadditive on disjoint vectors from the start i.e.

$$\|x + y\|^p \geq \|x\|^p + \|y\|^p \quad (x, y \in E, x \wedge y = 0)\ ,$$

then so is $\|\ \|_\triangledown$;

ii. *the expression*

$$\| x \|_{\vartriangle} := \sup \left\{ \left[\sum_1^n \| x_\iota \|^p \right]^{1/p} \ : \ n \in \mathbb{N} , \ \sum_1^n x_\iota = x \text{ disjoint} \right\} \quad (x \in E) ,$$

defines a Riesz quasi-norm on E that is equivalent to $\| \, \|$ *and* p*-superadditive on disjoint elements i.e.*

$$\| x + y \|_{\vartriangle}^p \geq \| x \|_{\vartriangle}^p + \| y \|_{\vartriangle}^p \quad (x, y \in E , x \wedge y = 0) .$$

Moreover, if $\| \, \|$ *was* p*-subadditive on disjoint vectors from the beginning, then so is* $\| x \|_{\vartriangle}$.

Conclusion: Both $\| \, \|_{\vartriangle\triangledown}$ *and* $\| \, \|_{\triangledown\vartriangle}$ *define* p*-additive Riesz quasi-norms on* E *that are equivalent to* $\| \, \|$.
In particular, using the generalization of Kakutani's theorem on abstract L^p*-spaces (theorem 1.2 page 46), there is a measure space* $(S, \mathcal{A}, \mu)$ *such that* E *is topologically Riesz isomorphic to* $L^p(S, \mathcal{A}, \mu)$.

Proof
By lemma 2.30 there is a $K \in [1, \infty)$ such that

$$\left. \begin{array}{l} \text{for all disjoint finite sequences } (x_\iota)_1^n \text{ in } E \ : \\ K^{-1} \, \| \sum_1^n x_\iota \| \ \leq \ \| (\, \| x_\iota \| \,)_1^n \|_p \ \leq K \, \| \sum_1^n x_\iota \| \ . \end{array} \right\} \qquad (*)$$

Moreover, since $p < \infty$ we have by the remark following lemma 2.30 that E has the principal projection property.
(i) From $(*)$ and the definition of $\| \, \|_{\triangledown}$ we obtain

$$K^{-1} \ \| \, \| \ \leq \ \| \, \|_{\triangledown} \leq K \ \| \, \| \ ,$$

for some $K > 0$, so that $\| \, \|_{\triangledown}$ is separating and quasi-subadditive.
Looking at its definition once more it is clear that $\| \, \|_{\triangledown}$ inherits absolute homogeneity from $\| \, \|$ as well as the properties

$$\| \, |x| \, \|_{\triangledown} = \| x \|_{\triangledown} \quad (x \in E) , \qquad (\dagger)$$

and

$$0 \leq u \leq v \text{ in } E \quad \Longrightarrow \quad \| u \|_{\triangledown} \leq \| v \|_{\triangledown} \ , \qquad (\ddagger)$$

which establishes that $\| \, \|_{\triangledown}$ is a Riesz quasi-norm equivalent to $\| \, \|$.

◁For $(\dagger)$ observe that if $\sum_1^n u_\iota = |x|$ disjoint, then $\sum_1^n x_\iota = x$ disjoint where

$$x_\iota := (x^+ \wedge u_\iota) - (x^- \wedge u_\iota)$$

satisfies $|x_\iota| = u_\iota$. To validate $(\ddagger)$ realize that if $\sum_1^n v_\iota = v$ disjoint then $\sum_1^n v_\iota \wedge u = u$ disjoint, and $0 \leq v_\iota \wedge u \leq v_\iota$.▷

Further, if x and y are disjoint then a disjoint composition of x and one of y can be concatenated to a disjoint decomposition of $x + y$ and hence

$$\| x + y \|_{\triangledown}^p \leq \| x \|_{\triangledown}^p + \| y \|_{\triangledown}^p \ .$$

Suppose finally that $\| \, \|$ is p-superadditive on disjoint vectors. Take $x, y \in E$ with $x \wedge y = 0$. If $x + y = \sum_1^n z_\iota$ disjoint, then $(x_\iota := z_\iota \wedge x)_1^n$ and $(y_\iota := z_\iota \wedge y)_1^n$ form disjoint decompositions of x and y respectively and $z_\iota = x_\iota + y_\iota$.

Thus, using p-superadditivity in the first inequality we see that

$$\textstyle\sum_1^n \|z_\iota\|^p = \sum_1^n \|x_\iota + y_\iota\|^p \geq \sum_1^n \|x_\iota\|^p + \sum_1^n \|y_\iota\|^p \geq \|x\|_\triangledown^p + \|y\|_\triangledown^p \,.$$

Hence, $\|x+y\|_\triangledown^p \geq \|x\|_\triangledown^p + \|y\|_\triangledown^p$ and thus $\|\,\|_\triangledown$ is p-superadditive on disjoint vectors as well.

(ii) The proof of the properties of $\|\,\|_\vartriangle$ is similar to that of *(i)* above except that the property

$$0 \leq u \leq v \text{ in } E \quad\Longrightarrow\quad \|u\|_\vartriangle \leq \|v\|_\vartriangle \,,$$

now ensues from the corresponding property of $\|\,\|$ *and* the principal projection property: if $\sum_1^n u_\iota = u$ disjoint then $\sum_1^n P_{u_\iota}(v) + (1 - P_u)(v) = v$ disjoint, and $0 \leq u_\iota \leq P_{u_\iota}(v)$ for all ι.

The conclusion follows by observing that if $\|\,\|'$ is a Riesz quasi-norm equivalent to $\|\,\|$, then $(*)$ holds with (another constant and) $\|\,\|$ replaced by $\|\,\|'$. □

Now we consider the case $p = \infty$, which is less transparent.

Theorem 2.32 (Characterization of infinite dimensional quasi-M-spaces)
Let $(E, \|\,\|)$ be an infinite dimensional quasi-Banach lattice such that every infinite orthonormal sequence is equivalent to the standard basis of c_0. Write "$\bigvee_1^n x_\iota = x$ disjoint" if $\bigvee_1^n x_\iota = x$ and $(x_\iota)_\iota$ is a disjoint sequence.
Then

$$\|x\|_{(\infty)} := \inf\{\, \textstyle\bigvee_1^n \|x_\iota\| \;:\; n \in \mathbb{N},\ \bigvee_1^n x_\iota = x \text{ disjoint}\,\} \in [0,\infty) \quad (x \in E)\,,$$

defines an equivalent Riesz quasi-norm on E such that

$$x \wedge y = 0 \quad\Longrightarrow\quad \|x+y\|_{(\infty)} = \|x\|_{(\infty)} \vee \|y\|_{(\infty)} \quad (x, y \in E)\,.$$

In terms of definition 2.33 below, $\|\,\|_{(\infty)}$ is an equivalent quasi-M-norm, and E is topologically Riesz isomorphic to a quasi-M-space.

Definition 2.33
A quasi-M-norm on a Riesz space E is a Riesz quasi-norm $\|\,\|$ on E such that

$$x \wedge y = 0 \quad\Longrightarrow\quad \|x+y\| = \|x \vee y\| = \|x\| \vee \|y\| \quad (x, y \in E)\,.$$

A quasi-M-space is a quasi-Banach lattice whose Riesz quasi-norm is a quasi-M-norm.

M-spaces are of course examples of quasi-M-spaces. In the next section we will, with partial success, go into the question whether quasi-M-spaces are M-spaces.

Proof of theorem 2.32
The proof that $\|\,\|_{(\infty)}$ is an equivalent Riesz quasi-norm is the same as that used for $\|\,\|_\triangledown$ of theorem 2.31. That $\|\,\|_{(\infty)}$ is Riesz then implies immediately that

$$\|x \vee y\|_{(\infty)} \geq \|x\|_{(\infty)} \vee \|y\|_{(\infty)} \quad (x, y \in E^+)\,.$$

Furthermore, if $x, y \in E$ with $x \wedge y = 0$, then a disjoint decomposition of x and one of y combine into a disjoint decomposition of $x \vee y$, so

$$\|x\|_{(\infty)} \vee \|y\|_{(\infty)} \geq \|x \vee y\|_{(\infty)} \,.$$

□

By combining the previous results, we obtain a joint characterization of L^p-spaces and quasi-M-spaces:

Theorem 2.34
For an infinite dimensional quasi-Banach lattice E, the following are equivalent:

(α) *E is topologically Riesz isomorphic to an L^p-space or to a quasi-M-space;*

(β) *every two infinite orthonormal sequences in E are equivalent;*

(γ) *the infinite orthonormal sequences in E are uniformly equivalent;*

(δ) *there is a constant K such that*

$$\left.\begin{array}{l} x_1,\dots,x_n \text{ disjoint}, \\ y_1,\dots,y_n \text{ disjoint}, \\ \|x_\iota\| = \|y_\iota\| \text{ all } i, \end{array}\right\} \quad \Rightarrow \quad \|\textstyle\sum_1^n x_\iota\| \le K\ \|\textstyle\sum_1^n y_\iota\| .$$

For the proof observe that $(\alpha) \Rightarrow (\gamma) \Rightarrow (\beta)$ is obvious, while $(\beta) \Rightarrow (\delta)$ is lemma 2.29 and $(\delta) \Rightarrow (\alpha)$ follows from 2.30, 2.31 and 2.32.

We conclude this section with the proof of lemma 2.29.

Proof of lemma 2.29
Instead of proving lemma 2.29 verbatim, we will prove something equivalent: we fix an orthonormal infinite sequence $(e_n)_1^\infty$ in E and show that

i. there is a constant $\alpha_E \in (0,\infty)$ such that

$$\left.\begin{array}{l} (u_\iota)_1^n \text{ orthonormal in } E\,, \\ (\lambda(i))_1^n \text{ sequence of scalars }, \end{array}\right\} \Rightarrow \alpha_E \left\|\sum_1^n \lambda(i) e_\iota\right\| \le \left\|\sum_1^n \lambda(i) u_\iota\right\| ;$$

(the orthonormal sequences in E are uniformly supervalent to $(e_n)_1^\infty$)

ii. there is a constant $\beta_E \in (0,\infty)$ such that

$$\left.\begin{array}{l} (u_\iota)_1^n \text{ orthonormal in } E\,, \\ (\lambda(i))_1^n \text{ sequence of scalars }, \end{array}\right\} \Rightarrow \left\|\sum_1^n \lambda(i) u_\iota\right\| \le \beta_E \left\|\sum_1^n \lambda(i) e_\iota\right\| .$$

(the orthonormal sequences in E are uniformly subvalent to $(e_n)_1^\infty$)

The properties we obtain from i. and ii. above if we replace E by a Riesz subspace $D \subset E$, will be referred to by saying that the orthonormal sequences in D are uniformly supervalent to $(e_n)_n$, and uniformly subvalent to $(e_n)_n$, respectively.

By topological completeness, every countable subset can be contained in a principal ideal (2.25). Therefore, to establish i. and ii., it suffices to show that, for every principal ideal $D \subset E$, the orthonormal sequences in D are both uniformly supervalent and uniformly subvalent to $(e_n)_n$.

$\triangleleft$Indeed, if e.g. the orthonormal sequences in E would not be uniformly supervalent to $(e_n)_1^\infty$, then there would be finite orthonormal sequences $(u_\iota^{[n]})_{\iota\in I_n}$ and corresponding sequences of scalars $\lambda^{[n]} = (\lambda^{[n]}(i))_{\iota\in I_n}$ such that

$$\left\|\textstyle\sum_{\iota\in I_n} \lambda^{[n]}(i) u_\iota^{[n]}\right\| \le n^{-1} \left\|\textstyle\sum_{\iota\in I_n} \lambda^{[n]}(i) e_\iota\right\| .$$

By topological completeness, those orthonormal sequences would already lie in a principal ideal (lemma 2.25 and so the orthonormal sequences in that principal ideal would already fail to be uniformly supervalent to $(e_n)_1^\infty$.$\triangleright$

Thus, from now on we focus our attention on a fixed principal ideal $D \subset E$ and prove i. and ii. with E replaced by D. Using lemma 2.25, we identify D with a $C(S)$ for some compact Hausdorff space S.

To prove that the orthonormal sequences in D are uniformly supervalent to $(e_n)_1^\infty$ we introduce for $U \subset S$ open:

$$D_U := \{f \in D \;:\; [|f| > 0] \subset U\} \qquad (\text{so } D_S = D)$$

and

$$\alpha(U) := \inf \left\{ \Big\| \sum_1^\infty \lambda(i) u_\iota \Big\| \;:\; \begin{array}{l} (u_\iota)_\iota \text{ orthonormal in } D_U\,, \\ \lambda \in c_{00}\,,\ \| \sum_1^\infty \lambda(i) e_\iota \| \geq 1 \end{array} \right\} \in [0, \infty)\,.$$

Then what we have to show is that $\alpha(S) > 0$. For that we employ the following lemma (whose proof is given on page 91 further on):

Lemma 2.35
If U is an open non-empty subset with $\alpha(U) = 0$, then for every $\varepsilon > 0$ there are disjoint open non-empty $U_1, U_2 \subset U$ with $\alpha(U_1) < \varepsilon$ and $\alpha(U_2) = 0$.

By lemma 2.35 we see that *if* $\alpha(S) = 0$, there exist disjoint open non-empty $S_1, S_2, \ldots \subset S$ such that $\alpha(S_k) < k^{-1}$. The latter, however, implies that there are disjoint finite orthonormal sequences $(u_\iota^{[k]})_{\iota=1}^{n_k}$ and scalars $(\lambda^{[k]}(i))_{\iota=1}^{n_k}$ satisfying

$$\Big\| \textstyle\sum_1^{n_k} \lambda^{[k]}(i) e_\iota \Big\| \geq 1\,, \text{while} \ \Big\| \textstyle\sum_1^{n_k} \lambda^{[k]}(i) u_\iota^{[k]} \Big\| < k^{-1}\,.$$

Concatenate the sequences $(u_\iota^{[k]})_{\iota=1}^{n_k}$, $k = 1, 2, \ldots$, into one orthonormal *infinite* sequence $(u_\jmath)_\jmath$ and the scalars $(\lambda^{[k]}(i))_{\iota=1}^{n_k}$, $k = 1, 2, \ldots$ into an infinite sequence of scalars $(\lambda(j))_\jmath$ such that

$$\| \textstyle\sum_{n_1+\ldots+n_{k-1}+1}^{n_1+\ldots+n_k} \lambda(j) u_\jmath \| = \| \sum_1^{n_k} \lambda^{[k]}(i) u_\iota^{[k]} \| < k^{-1} \quad (k \in \mathbb{N})\,.$$

However, the equivalence of orthonormal infinite sequences implies that $(e_n)_n$ is perfectly homogeneous, in particular there is a $M \in (0, \infty)$ such that

$$M^{-1} \| \textstyle\sum_{\iota=1}^{n_k} \lambda^{[k]}(i) e_\iota \| \leq \| \sum_{n_1+\ldots+n_{k-1}+1}^{n_1+\ldots+n_k} \lambda(j) e_\jmath \|$$

(lemma 2.19 on page 79). Thus $(u_\iota)_\iota$ and $(e_\iota)_\iota$ are *not* equivalent: *contradiction* (with the assumed equivalence of infinite orthonormal sequences in E).

Subvalence of the finite orthonormal sequences in D to $(e_n)_1^\infty$ is proved along the same lines. We set for $U \subset S$ open:

$$\beta(U) := \sup \left\{ \Big\| \sum_1^\infty \lambda(i) u_\iota \Big\| \;:\; \begin{array}{l} (u_\iota)_\iota \text{ orthornormal in } D_U\,, \\ \lambda \in c_{00}\,,\ \| \sum_1^\infty \lambda(i) e_\iota \| \leq 1 \end{array} \right\} \in (0, \infty]\,,$$

where D_U is as in the definition of $\alpha(U)$ above, and we want to prove that $\beta(S) < \infty$. For the latter we now rely on

Lemma 2.36
If U is an open non-empty subset with $\beta(U) = \infty$, then for every $M > 0$ there are disjoint open non-empty $U_1, U_2 \subset U$ with $\beta(U_1) > M$ and $\beta(U_2) = \infty$.

(The proof of 2.36 follows on page 92).

Now, lemma 2.36 implies that *if* $\beta(S) = \infty$, there exist disjoint open non-empty $S_1, S_2, \ldots \subset S$ such that $\beta(S_k) > k$, from which we deduce as above the existence of an infinite orthonormal sequence in D that is not equivalent to $(e_n)_1^\infty$: *contradiction.*

Apart from the pending proofs of lemmata 2.35 and 2.36 we have now established lemma 2.29. □

The ideas of the proofs of lemmata 2.35 and 2.36 are most transparent in the special case that $\rfloor\!\rfloor\,\lfloor\!\lfloor$ is a norm, and S is basically disconnected. This special case occurs e.g. if the orthonormal sequences in E are equivalent to the standard basis of ℓ^p for a $p \in [1, \infty)$ for in that case D is σ-lateral complete (see the remark preceding theorem 2.31) and hence S is basically disconnected (see 0.82).

To get a feeling of the main line of argument we will treat the special case mentioned above first, and generalize thereafter.

Proof of lemma 2.35 in case $\rfloor\!\rfloor\,\lfloor\!\lfloor =: \|\ \|$ is a norm and S is basically disconnected

First observe

$$\left.\begin{array}{l}\text{(Case } \rfloor\!\rfloor\,\lfloor\!\lfloor =: \|\ \| \text{ is a norm and } S \text{ is basically disconnected):}\\ \text{If } U_1, U_2 \text{ are clopen subsets of } S, \text{ then } \alpha(U_1) \wedge \alpha(U_2) \le 4\alpha(U_1 \cup U_2).\end{array}\right\}(\#')$$

◁For the proof of $(\#')$ let U_1, U_2 be clopen in S. Take a finite orthonormal sequence $(u_\iota)_{\iota \in I}$ in $D_{U_1 \cup U_2}$ (where $I \subset \mathbb{N}$), and a $\lambda \in c_{00}$ such that $\|\sum_1^\infty \lambda(i) e_\iota\| \ge 1$.

For each i we write $u_\iota = u_{\iota 1} + u_{\iota 2}$ with $u_{\iota j} \in D_{U_j}$ (e.g. $u_{\iota 1} = u\mathbb{1}_{U_1}$ and $u_{\iota 2} = u - u_{\iota 1}$).

By subadditivity of $\|\ \|$, we have for each i: $\|u_{\iota 1}\| \ge 1/2$ or $\|u_{\iota 2}\| \ge 1/2$. Thus $I = I_1 \cup I_2$ with $I_1 := \{i : \|u_{\iota 1}\| \ge 1/2\}$ and $I_2 := I \setminus I_1 \subset \{i : \|u_{\iota 2}\| \ge 1/2\}$.

Using subadditivity again, either $\|\sum_1^\infty \lambda \mathbb{1}_{I_1}(\iota) e_\iota\| \ge 1/2$ or $\|\sum_1^\infty \lambda \mathbb{1}_{I_2}(\iota) e_\iota\| \ge 1/2$.

Say the first is the case. Then $I_1 \ne \emptyset$ and with $\hat{u}_{\iota 1} := u_{\iota 1} / \|u_{\iota 1}\|$ $(i \in I_1)$ and with $\lambda_1 := \lambda \mathbb{1}_{I_1} / \|\sum_1^\infty \lambda \mathbb{1}_{I_1}(i) e_\iota\|$ we have $0 \le \hat{u}_{\iota 1} \le 2u_{\iota 1}$ and $|\lambda_1(\iota)| \le 2|\lambda(\iota)|$, so that

$$\left\|\sum_{\iota \in I_1} \lambda_1(i) \hat{u}_{\iota 1}\right\| \le \left\|\sum_{\iota \in I_1} 2\lambda(\iota) \cdot 2u_{\iota 1}\right\| \le 4 \left\|\sum_{\iota \in I} \lambda(\iota) u_\iota\right\| .$$

Ergo,

$$\alpha(U_1) \wedge \alpha(U_2) \le \alpha(U_1) \le \left\|\sum_{\iota \in I_1} \lambda_1(i) u_{\iota 1}\right\| \le 4 \left\|\sum_{\iota \in I} \lambda(i) u_\iota\right\| .$$

In the other case ($\|\sum_1^\infty \lambda \mathbb{1}_{I_2}(i) e_\iota\| \ge 1/2$), we reach a similar conclusion using a similar argument.

The claim now follows by taking the infimum over all orthonormal sequences $(u_\iota)_{\iota \in I}$ in $D_{U_1 \cup U_2}$ and $\lambda \in c_{00}$ such that $\|\sum_1^\infty \lambda(i) e_\iota\| \ge 1$.▷

Using $(\#')$ we prove $(\flat')$ below, which establishes lemma 2.35.

$$\left.\begin{array}{l}\text{(Case } \rfloor\!\rfloor\,\lfloor\!\lfloor =: \|\ \| \text{ is a norm and } S \text{ is basically disconnected)}\\ \text{Let } U \text{ be a clopen, non-empty subset of } S \text{ such that } \alpha(U) = 0.\\ \text{Suppose } 0 < \varepsilon < 1/3.\\ \text{Then there exist non-empty, clopen, disjoint subsets } U_1, U_2 \subset U \text{ such}\\ \text{that } \alpha(U_1) < 3\varepsilon \text{ and } \alpha(U_2) = 0.\end{array}\right\}(\flat')$$

◁For a proof of $(\flat')$ take an orthonormal sequence $(u_\iota)_{\iota \in I}$ in D_U, and a $\lambda \in c_{00}$ (say $\lambda(i) = 0$ for $i > n$) such that $\|\sum_1^n \lambda(i) u_\iota\| < \varepsilon$, and $\|\sum_1^n \lambda(i) e_\iota\| > 1$.

For each i we have: $\|\lambda(i) e_\iota\| = |\lambda(i)| = \|\lambda(i) u_\iota\| \le \|\sum_1^n \lambda(i) u_\iota\| < \varepsilon < 1/3$, so by subadditivity of the norm:

$$\left\|\sum_1^{m+1} \lambda(i)e_i\right\| \le \|\sum_1^m \lambda(i)e_i\| + 1/3 \quad (\text{all } m)\,.$$

As a result there is a $1 < k < n$ with

$$1/3 < \left\|\sum_1^k \lambda(i)e_i\right\| \le 2/3\,,$$

and using subadditivity again:

$$\|\sum_{k+1}^n \lambda(i)e_i\| \ge \|\sum_1^n \lambda(i)e_i\| - \left\|\sum_1^k \lambda(i)e_i\right\| > 1 - 2/3 = 1/3$$

Setting

$$\lambda_1 := \lambda \cdot \mathbb{1}_{\{1,\dots,k\}} / \| \sum_1^k \lambda(i)e_i \|\,, \quad \lambda_2 := \lambda \cdot \mathbb{1}_{\{k+1,\dots,n\}} / \| \sum_{k+1}^n \lambda(i)e_i \|$$

we have $\|\sum_1^\infty \lambda_1(i)e_i\| = \|\sum_1^\infty \lambda_2(i)e_i\| = 1$, while

$$\|\sum_1^n \lambda_1(i)u_i\| \le 3\,\|\sum_1^n \lambda(i)u_i\| < 3\varepsilon\,, \quad \text{and likewise} \quad \|\sum_1^n \lambda_2(i)u_i\| < 3\varepsilon\,.$$

Put

$$V := \overline{[\sum_1^k u_i > 0]}\,, \quad \text{and} \quad W := U \setminus V\,,$$

then V, W are clopen, non-empty, disjoint, while both $\alpha(V)$ and $\alpha(W)$ are smaller than 3ε. By (#′), one of $\alpha(V)$ and $\alpha(W)$ has to be 0, which we can take as U_2. The remaining one satisfies the condition for U_1.▷

□

Now we look at the

Proof of lemma 2.36 in case $\rfloor\!\!\lfloor\ =: \|\ \|$ is a norm and S is basically disconnected

We start with observing that

$$\left.\begin{array}{l}\text{(Case } \rfloor\!\!\lfloor\ =: \|\ \| \text{ is a norm and } S \text{ is basically disconnected)}\\ \text{If } U_1, U_2 \text{ are clopen subsets of } S\text{, then } \beta(U_1 \cup U_2) \le \beta(U_1) + \beta(U_2).\end{array}\right\}(\Diamond')$$

◁For the proof of $(\Diamond')$ let U_1, U_2 be clopen subsets of S. Take a finite orthonormal sequence $(u_i)_{i\in I}$ in $D_{U_1 \cup U_2}$, and $\lambda \in c_{00}$ such that $\|\sum_1^\infty \lambda(i)e_i\| \le 1$. For each i, we write $u_i = u_{1i} + u_{2i}$ with $u_{1i} \in D_{U_1}$ and $u_{2i} \in D_{U_2}$. Let $I_1 := \{i : u_{1i} \ne 0\}$, $I_2 := \{i : u_{2i} \ne 0\}$, and set for $i \in I_1$ and $i \in I_2$ respectively: $\hat{u}_{1i} := u_{1i}/\ \|u_{1i}\|$ and $\hat{u}_{2i} := u_{2i}/\ \|u_{2i}\|$.
Subadditivity then implies

$$\begin{aligned}\left\|\sum_{i\in I} \lambda(i)u_i\right\| &= \left\|\sum_{i\in I_1} \lambda(i)u_{1i} + \sum_{i\in I_2} \lambda(i)u_{2i}\right\| \le \left\|\sum_{i\in I_1} \lambda(i)u_{1i}\right\| + \left\|\sum_{i\in I_2} \lambda(i)u_{2i}\right\| \\ &\le \left\|\sum_{i\in I_1} \lambda(i)\hat{u}_{1i}\right\| + \left\|\sum_{i\in I_2} \lambda(i)\hat{u}_{2i}\right\| \le \beta(U_1) + \beta(U_2)\,.\end{aligned}$$

where we use in the second inequality that $0 < \|u_{ki}\| \le 1$ (all k and i), and in the last inequality that $\left\|\sum_{i\in I_k} \lambda(i)e_i\right\| \le \|\sum_{i\in I} \lambda(i)e_i\| \le 1$ (all k).
The claim now follows by taking the supremum over all finite orthonormal sequences $(u_i)_{i\in I}$ in $D_{U_1\cup U_2}$, and $\lambda \in c_{00}$ such that $\|\sum_1^\infty \lambda(i)e_i\| \le 1$.▷

From $(\Diamond')$ we obtain (∂') below, which proves lemma 2.36.

$$\left.\begin{array}{l}\text{(Case } \rfloor\!\!\lfloor \text{ is a norm and } S \text{ is basically disconnected)}\\ \text{Let } U \text{ be a clopen, non-empty subset of } S \text{ such that } \beta(U) = \infty\text{, and}\\ \text{suppose } 1 < M < \infty.\\ \text{Then there exist non-empty, clopen, disjoint subsets } U_1, U_2 \subset U \text{ such}\\ \text{that } \beta(U_1) > M \text{ and } \beta(U_2) = \infty.\end{array}\right\}(\partial')$$

$\lhd$For a proof, take an orthonormal sequence $(u_\iota)_{\iota\in I}$ in D_U, and a $\lambda \in c_{00}$ (say $\lambda(\iota) = 0$ for $i > n$) such that

$$\|\textstyle\sum_1^n \lambda(i)e_\iota\| \le 1 \text{ and } \|\sum_1^n \lambda(i)u_\iota\| > 2M + 1 .$$

Observe that the first inequality implies that $\|\lambda(i)u_\iota\| = |\lambda(i)| \le 1$ (all i), so that we can find, as in the proof of ($\flat'$) above, a k such that $1 < k < n$ and

$$M < \left\|\textstyle\sum_1^k \lambda(i)u_\iota\right\| \le M + 1 , \text{ and therefore } \|\sum_{k+1}^n \lambda(i)u_\iota\| > M .$$

Further, $\left\|\sum_1^k \lambda(i)e_\iota\right\| \le 1$ and $\|\sum_{k+1}^n \lambda(i)e_\iota\| \le 1$, so that with

$$V := \overline{[\textstyle\sum_1^k u_\iota > 0]} , \text{ and } W := U \setminus V ,$$

both $\beta(V)$ and $\beta(W)$ are larger than M, while one of them has to be ∞ (using ($\Diamond'$)). Choose U_1 and U_2 accordingly.$\rhd$ □

For proving the lemmata 2.35 and 2.36 in the general case, we have to argue more carefully: $\|\|$ may not be subadditive and S may not be basically disconnected in general.

The first is not essential: by passing to an equivalent quasi-norm if necessary, we may (and will) assume that $\|\|$ is r-subadditive for some $r \in (0, 1]$.

The second obstacle, the possible lack of clopen sets, has to be bypassed at two points. At one of them we use an approximation argument that we will discuss in due course. The other point is the following: given two open (but not necessarily clopen) subsets U_1, U_2, and a function u with support in $U_1 \cup U_2$, we want to decompose $u = u_1 + u_2$ with each u_ι having its support in U_ι. In the proof of ($\#'$) and ($\Diamond'$) this was simple, because one of U_1 and U_2 was clopen. For the general case the following turns out to be sufficient:

Lemma 2.37

Let X be a compact Hausdorff space, let $U \subset X$ open, and let $f \in C(X)^+$ with $[f > 0] \subset U$.

Suppose that $w_1, w_2 \in C(X)^+$ are such that $U \subset [w_1 > 0] \cup [w_2 > 0]$.

Then there are $f_1, f_2 \in C(X)^+$ such that

$$\begin{cases} f = f_1 + f_2, \\ [f_1 > 0] \subset U \cap [w_1 > 0] \\ [f_2 > 0] \subset U \cap [w_2 > 0] \end{cases}$$

$\lhd$For a proof of this lemma observe that

$$f_\iota(x) := \begin{cases} \dfrac{f(x)w_\iota(x)}{w_1(x) + w_2(x)} & \text{if } x \in U , \\ 0 & \text{if } x \in X \setminus U . \end{cases} \qquad (i = 1, 2 ; x \in X) ,$$

is well-defined, net-continuous at points of U (since U is open), and net-continuous at points of $X \setminus U$ (since $|f_\iota| \le |f|$).$\rhd$

In the proofs of lemmata 2.35 and 2.36 *for the general case*, we will follow the line of argument as exposed in the special cases above as much as possible.

By the same arguments as discussed in the proof of the "only-if" part of Zippin's theorem (p. 77) it suffices to prove lemma 2.29 (and thus 2.35 and 2.36) for an equivalent Riesz quasi-norm which we take to be r-subadditive ($r \in (0, 1]$) (0.122).

Proof of lemma 2.35 in the general case

Instead of (#′) we have in the general case

> *Let U be an open subset of S and suppose that there are $w_1, w_2 \in C(S)^+$ such that $U \subset [w_1 > 0] \cup [w_2 > 0]$.*
> *Then, with $U_i := U \cap [w_i > 0]$ $(i = 1, 2)$:*
> $$\alpha(U_1) \wedge \alpha(U_2) \leq 4^{1/r} \alpha(U_1 \cup U_2) .$$
> (#)

The proof of (#) is like that for the special case (#′) above, except for two points. First, we use r-subadditivity instead of subadditivity (replace $\| \;\|$ by $\rrbracket \llbracket^r$); secondly, for the splitting of each u_i as $u_i = u_{i1} + u_{i2}$ with u_{i1} having support in U_1 and u_{i2} having its support in U_2 we use the lemma 2.37 above.

Not surprisingly, (#) preludes an analogue of $(\flat)'$:

> *Let U be an open, non-empty subset of S such that $\alpha(U) = 0$.*
> *Suppose $0 < \varepsilon < (1/3)^{1/r}$.*
> *Then there exist non-empty, open, disjoint subsets $U_1, U_2 \subset U$ such that $\alpha(U_1) < 3^{1/r}\varepsilon$ and $\alpha(U_2) = 0$.*
> $(\flat)$

After replacing subadditivity by r-subadditivity $(\| \;\| \mapsto \rrbracket \llbracket^r)$ the proof of $(\flat)$ runs as the proof of the special case $(\flat')$ above up to the point where we define V and W i.e. we obtain an orthonormal sequence $(u_i)_{i \in I}$ in D_U, and $\lambda_1, \lambda_2 \in c_{00}$ and $k, n \in \mathbb{N}$ such that $\rrbracket \sum_1^\infty \lambda_1(i) e_i \llbracket = \rrbracket \sum_1^\infty \lambda_2(i) e_i \llbracket = 1$, $[|\lambda_1| > 0] \subset \{1, \ldots, k\}$, $[|\lambda_2| > 0] \subset \{k+1, \ldots, n\}$, and

$$\rrbracket \textstyle\sum_1^n \lambda_1(i) u_i \llbracket < 3^{1/r}\varepsilon , \quad \rrbracket \sum_1^n \lambda_2(i) u_i \llbracket < 3^{1/r}\varepsilon .$$

To obtain two disjoint open subsets $U_1, U_2 \subset U$ as announced, we use an approximation argument. For $\delta \in (0,1)$ set

$$u_i^\delta := (u_i - \delta \mathbb{1})^+ \quad (i \in I) .$$

Then for every i: $0 \leq u_i^\delta \leq u_i$ (so $u_i^\delta \wedge u_j^\delta = 0$ if $i \neq j$) and

$$\hat{u}_i^\delta := u_i^\delta / \left\rrbracket u_i^\delta \right\llbracket \to u_i \quad \text{uniformly in } C(S) \text{ if } \delta \downarrow 0 .$$

As a result,

$$\lim_{\delta \downarrow 0} \rrbracket \textstyle\sum_1^\infty \lambda_1(i) \hat{u}_i^\delta \llbracket = \rrbracket \sum_1^\infty \lambda_1(i) u_i \llbracket < 3^{1/r}\varepsilon ,$$
$$\lim_{\delta \downarrow 0} \rrbracket \textstyle\sum_1^\infty \lambda_2(i) \hat{u}_i^\delta \llbracket = \rrbracket \sum_1^\infty \lambda_2(i) u_i \llbracket < 3^{1/r}\varepsilon ,$$

for which we have used that $\rrbracket \llbracket$ is continuous with respect to $\| \;\|_\infty$:

$$| \rrbracket f \llbracket^r - \rrbracket g \llbracket^r | \leq \rrbracket f - g \llbracket^r \leq \|f - g\|_\infty^r \rrbracket \mathbb{1} \llbracket^r \quad (f, g \in C(S)) .$$

Take now a $\delta \in (0,1)$ such that

$$\left\rrbracket \textstyle\sum_1^\infty \lambda_1(i) \hat{u}_i^\delta \right\llbracket < 3^{1/r}\varepsilon , \text{ and } \left\rrbracket \sum_1^\infty \lambda_2(i) \hat{u}_i^\delta \right\llbracket < 3^{1/r}\varepsilon ,$$

then

$$\alpha\left([\textstyle\sum_1^k u_i > \delta]\right) = \alpha\left([\sum_1^k u_i^\delta > 0]\right) < 3^{1/r}\varepsilon$$

and

$$\alpha\left([\textstyle\sum_{k+1}^n u_i > \delta]\right) = \alpha\left([\sum_{k+1}^n u_i^\delta > 0]\right) < 3^{1/r}\varepsilon .$$

Since $\alpha(T') \le \alpha(T)$ if $T' \supset T$ we conclude:

$$\alpha\Big(\underbrace{[\textstyle\sum_{k+1}^n u_\iota < \delta] \cap U}_{=:V}\Big) \le \alpha\Big([\textstyle\sum_1^k u_\iota > \delta]\Big) < 3^{1/r}\varepsilon\,,$$

and

$$\alpha\Big(\underbrace{[\textstyle\sum_1^k u_\iota < \delta] \cap U}_{=:W}\Big) \le \alpha\Big([\textstyle\sum_{k+1}^n u_\iota > \delta]\Big) < 3^{1/r}\varepsilon\,,$$

but by (#), one of $\alpha(V)$ and $\alpha(W)$ has to be zero. In the first case, we choose

$$U_1 := [\textstyle\sum_{k+1}^n u_\iota > \delta]\,, \qquad U_2 := V = [\textstyle\sum_{k+1}^n u_\iota < \delta] \cap U\,,$$

and in the second case we take

$$U_1 := [\textstyle\sum_1^k u_\iota > \delta]\,, \qquad U_2 := W = [\textstyle\sum_1^k u_\iota < \delta] \cap U\,.$$

□

Proof of lemma 2.36 in the general case
The analogue of $(\lozenge')$ we use in the general case is

> *Let U be an open subset of S and suppose that there exist $w_1, w_2 \in C(S)^+$ such that $U \subset [w_1 > 0] \cup [w_2 > 0]$.*
> *Then, with $U_\iota := U \cap [w_\iota > 0]$ $(i = 1, 2)$:*
> $$\beta(U)^r \le \beta(U_1)^r + \beta(U_2)^r\,. \qquad (\lozenge)$$

Its proof can be obtained from that of $(\lozenge')$ by replacing the subadditivity of the norm $\|\;\|$ by r-subadditivity of the quasi-norm $[\![\;]\!]$ and using for the splitting of each u_ι lemma 2.37.
Subsequently, from $(\lozenge)$ we obtain (∂) below.

> *Let U be an open, non-empty subset of S such that $\beta(U) = \infty$, and suppose $1 < M < \infty$.*
> *Then there exist non-empty, open, disjoint subsets $U_1, U_2 \subset S$ such that $\beta(U_1) > M$ and $\beta(U_2) = \infty$.* $\qquad (\partial)$

For a proof, observe that there is an orthonormal sequence $(u_\iota)_{\iota\in I}$ in D_U, and a $\lambda \in c_{00}$ (say $\lambda(i) = 0$ for $i > n$) such that

$$[\![\textstyle\sum_1^\infty \lambda(i)e_\iota]\!] \le 1 \text{ and } [\![\textstyle\sum_1^n \lambda(i)u_\iota]\!]^r > 2M^r + 1\,,$$

Using r-subadditivity instead of subadditivity $(\|\;\| \mapsto [\![\;]\!]^r)$, as in the special case (∂') above we find a $1 < k < n$ such that

$$[\![\textstyle\sum_1^k \lambda(i)u_\iota]\!]^r > M^r \text{ and } [\![\textstyle\sum_{k+1}^n \lambda(i)u_\iota]\!]^r > M^r\,.$$

To find disjoint open $U_1, U_2 \subset U$ as required, we employ an approximation argument as in the proof of $(\flat)$, using $(\lozenge)$ instead of (#). □

Remarks 2.38

Ad theorem 2.34 We can also formulate theorem 2.34 in more topological terms. To this end, we introduce the following.

Definition
Let E be a locally bounded topological vector space, and let $(x_\iota)_1^\infty$ be a sequence in E. We say that $(x_\iota)_1^\infty$ is bounded away from 0 and ∞ if there are a bounded zero-neighborhood V and $\alpha < \beta$ in $(0, \infty)$ such that $\{x_\iota\}_{\iota\in\mathbb{N}} \subset \beta V$ and $\{x_\iota\}_{\iota\in\mathbb{N}} \subset E \setminus \alpha V$. In other words, if p is the Minkowski-functional of V (p. 30), then $\alpha \leq p(x_\iota) \leq \beta$ for all $\iota \in \mathbb{N}$.

The implication of e.g. (α) by (γ) in theorem 2.34 (p. 89) then reads:

Theorem
Let E be a locally solid, locally bounded Riesz space such that every two disjoint sequences that are bounded away from 0 and ∞, are equivalent.
Then E is topologically Riesz isomorphic to an L^p-space or to a quasi-M-space.

For the proof use the theory developed in 0.122 (p. 39) to equip E with a Riesz quasi-norm $\|\!|\,\|\!|$ that generates the topology; the assumption then implies that all $\|\!|\,\|\!|$-orthonormal sequences are equivalent, and so the occasion arises to apply theorem 2.34 $(\gamma) \Rightarrow (\alpha)$.

Ad lemma 2.35 We took $\|\sum_1^n \lambda(\iota)e_\iota\| > 1$ (instead of ≥ 1) in $(\mathfrak{b}')$ to have $\|\!|\sum_1^n \lambda_j(\iota)e_\iota\|\!| < 1/3$ $(j = 1, 2)$ for the approximation argument in $(\mathfrak{b})$.

Ad the use of Zippin's theorem For proving theorem 2.34 we only needed a generalization of Zippin's theorem in the context of *Riesz* quasi-norms. Since Riesz quasi-norms are automatically absolutely monotone with respect to a basis, proposition 2.14 (p. 75) was not strictly necessary.

2.5 Quasi-M-spaces versus M-spaces

In this section we study the question whether a quasi-M-space is an M-space. In some special cases we can give a positive answer, but in general we must leave the question open: we have not found a proof nor a counterexample. Together with well-known characterizations of M-spaces (as Banach lattices), our results do allow us to obtain two generalizations of a joint characterization of L^p-spaces, $p \in [1, \infty)$ and $c_0(X)$, X discrete, found in [LiTz2, theorem 1.b.12, p. 22]:

Corollary 2.46 *A Banach lattice is topologically Riesz isomorphic to an L^p-space for some $p \in [1, \infty)$ or to an M-space if and only if every two orthonormal sequences are equivalent.*

Corollary 2.43 *If $(E, \|\!|\,\|\!|)$ is a quasi-Banach lattice such that every two orthonormal sequences are equivalent and $\|\!|\,\|\!|$ is weak-Fatou, then E is topologically Riesz isomorphic to an L^p-space for some $p \in (0, \infty)$ or to an M-space.*
As a special case: A quasi-Banach lattice in which all orthonormal sequences are equivalent has an order continuous quasi-norm, if and only if E is topologically Riesz isomorphic to an L^p-space, for some $p \in (0, \infty)$ or to a $c_0(X)$, for some discrete set X.

To avoid confusion about the meaning of the prefix M in different terms, we begin with an explicit listing of the notions concerned.

Convention 2.39

An M-(semi)norm on E is a Riesz (semi-)norm $\|\ \|$ with the M-property: if u, v are positive, then $\|u \vee v\| = \|u\| \vee \|v\|$. An M-space is a Banach lattice with an M-norm.

A quasi-M-norm is a Riesz quasi-norm $|\!|\!|\ |\!|\!|$ that is ∞-additive: if u, v are disjoint and positive, then $|\!|\!|u \vee v|\!|\!| = |\!|\!|u|\!|\!| \vee |\!|\!|v|\!|\!|$. A quasi-$M$-space is a quasi-Banach lattice with a quasi-M-norm.

Our approach to the problem whether a quasi-M-space is topologically Riesz isomorphic to an M-space starts with the following observation:

A quasi-M-space $(E, |\!|\!|\ |\!|\!|)$ is topologically Riesz isomorphic to an M-space if and only if there exists an M-norm $\|\ \|$ on E that is equivalent to $|\!|\!|\ |\!|\!|$.

The following lemma singles out a natural candidate for that M-norm:

Lemma 2.40

Let E be an Archimedean Riesz space and let $|\!|\!|\ |\!|\!| : E \to [0, \infty)$ be r-subadditive, absolutely homogeneous and Riesz (e.g. an r-subadditive Riesz quasi-norm).
Then the formula

$$|\!|\!|x|\!|\!|_\vee := \inf\left\{ \bigvee_1^n |\!|\!|u_\iota|\!|\!| \ : n \in \mathbb{N},\, u_\iota \in E^+,\, \bigvee_1^n u_\iota = |x| \right\}$$

defines the greatest M-semi-norm below $|\!|\!|\ |\!|\!|$.

If D is a principal ideal of E and $\hat{}: D \to C(X)$ is a Yoshida representation of D, then (the restriction to D of) $|\!|\!|\ |\!|\!|_\vee$ is represented as

$$|\!|\!|f|\!|\!|_\vee = \|\hat{f} \cdot \phi\|_\infty \quad (f \in D),$$

where

$$\phi(x) := \inf\{ |\!|\!|e|\!|\!| : e \in D^+, 0 \le \hat{e} \le \mathbb{1}, \hat{e}(x) = 1\}.$$

Proof

We first make some elementary observations:

(0) For all $f \in E$: $|\!|\!|f|\!|\!|_\vee \le |\!|\!|f|\!|\!|$.

(1) For all $0 \le u, v \in E$: $|\!|\!|u \vee v|\!|\!|_\vee = |\!|\!|u|\!|\!|_\vee \vee |\!|\!|v|\!|\!|_\vee$.

Combined with the fact that $|\!|\!|\,|x|\,|\!|\!|_\vee = |\!|\!|x|\!|\!|_\vee$, this implies that $|\!|\!|\ |\!|\!|_\vee$ is Riesz.

Indeed, for the inequality $\le$ observe that if $u = \vee_1^n u_\iota$ and $v = \vee_1^m v_j$, then we can write $u \vee v = \vee_1^{n+m} w_k$ where $(w_k)_k$ is the concatenation of $(u_\iota)_\iota$ and $(v_j)_j$.
The converse inequality follows from the fact that if $u \vee v = \vee_1^n w_\iota$, then certainly $u = \vee_1^n (w_\iota \wedge u)$ and $v = \vee_1^n (w_\iota \wedge v)$.

(2) If D is a (principal) ideal of E and $\|\ \|$ is a M-semi-norm on D such that $\|\ \| \le |\!|\!|\ |\!|\!|$ on D, then $\|\ \| \le |\!|\!|\ |\!|\!|_\vee$ on D.

In particular: $|\!|\!|\ |\!|\!|_\vee$ is greater than every M-semi-norm below $|\!|\!|\ |\!|\!|$.

Indeed, if D and $\|\ \|$ are as in the premiss, then

$$\begin{aligned} |\!|\!|x|\!|\!|_\vee &= \inf\{\vee_1^n |\!|\!|u_\iota|\!|\!| \ : u_\iota \in D,\ \vee_1^n u_\iota = |x|\} \\ &\ge \inf\{\vee_1^n \|u_\iota\| \ : u_\iota \in D,\ \vee_1^n u_\iota = |x|\} = \|x\|. \end{aligned}$$

From the above observations we establish the local representation of $|\!|\!|\ |\!|\!|_\vee$ from which the remaining claims of lemma 2.40 follow.

Let D be a principal ideal of E. We identify D with a norm-dense, $\mathbb{1}$-containing Riesz subspace of $C(X)$ and we omit the symbol $\hat{}$.

(3) For all $f \in D$: $\|f\phi\|_\infty \le \|f\|$,

Therefore, by (2), for all $f \in D$: $\|f\phi\|_\infty \le \|f\|_\vee \quad (f \in D)$.

We prove that

$$u(x)\phi(x) \le \|u\| \quad (x \in X,\ u \in D^+),$$

which implies (3) since both $\|\ \|_\infty$ and $\|\ \|$ are Riesz. Let $u \in D^+$ and $x \in X$. If $u(x) = 0$, the inequality trivially holds. Otherwise,

$$u/u(x) \wedge \mathbb{1} \in \{e \in D : 0 \le e \le \mathbb{1},\ e(x) = 1\},$$

and thus $\phi(x) \le \|u/u(x) \wedge \mathbb{1}\| \le \|u/u(x)\| = \|u\| / u(x)$ by definition.

(4) For all $f \in D$: $\|f\phi\|_\infty \ge \|f\|_\vee$.

Let $u \in D^+$, and let $\varepsilon, \beta > 0$. Using a compactness argument and the definition of ϕ we can find $x_1, \ldots, x_n \in X$, $e_{x_1}, \ldots, e_{x_n} \in D^+$ with

$$0 \le e_{x_i} \le \mathbb{1}, \qquad e_{x_i}(x_i) = 1, \qquad \|e_{x_i}\| \le \phi(x_i) + \varepsilon \quad (i \in \{1, \ldots, n\}).$$

such that the sets

$$[\,u(x_i)e_{x_i} + \beta\mathbb{1} > u\,] \quad (i \in \{1, \ldots, n\})$$

cover X. Then

$$u \le \bigvee_1^n \left(u(x_i)e_{x_i} + \beta\mathbb{1}\right) \quad \text{in } D^+.$$

That, together with the r-subadditivity and absolute homogeneity of $\|\ \|$ yields

$$\begin{aligned}
(\|u\|_\vee)^r &\underset{(1)}{\le} \Big(\bigvee_1^n \|u(x_i)e_{x_i} + \beta\mathbb{1}\|_\vee\Big)^r \underset{(0)}{\le} \Big(\bigvee_1^n \|u(x_i)e_{x_i} + \beta\mathbb{1}\|\Big)^r \\
&\le \bigvee_1^n \|u(x_i)e_{x_i}\|^r + \|\beta\mathbb{1}\|^r = \bigvee_1^n u(x_i)^r\,\|e_{x_i}\|^r + \beta^r\,\|\mathbb{1}\|^r \\
&\le \|u \cdot (\phi + \varepsilon\mathbb{1})\|_\infty^r + \beta^r\,\|\mathbb{1}\|^r .
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ and $\beta \downarrow 0$ establishes (4). □

As a corollary we obtain a necessary and sufficient condition for a Riesz quasi-norm to be (equivalent to) an M-norm.

Corollary 2.41

Let $(E, \|\ \|)$ be a quasi-normed Riesz space.

If

$$\|u \vee v\| = \|u\| \vee \|v\| \quad (u, v \in E^+), \qquad \textbf{(}M\textbf{-property)}$$

then $\|\ \|$ is an M-norm.

More generally, if

$$\sup\{\ \|\textstyle\bigvee_1^n u_i\| \ :\ u_i \in E^+,\ \|u_i\| \le 1\} < \infty$$

i.e.

there is a constant $K \in [1, \infty)$ such that

$$\|\textstyle\bigvee_1^n u_i\| \le K\ \bigvee_1^n \|u_i\| \quad (u_1, \ldots, u_n \in E^+), \qquad \textbf{(weakened } M\textbf{-property)}$$

then there is an M-norm on E that is equivalent to $\|\ \|$.

We now exploit the above corollary to find some sufficient conditions on a quasi-M-space to be topologically Riesz isomorphic to an M-space.

If $\|\ \| : E \to [0,\infty)$ is (equivalent to) a quasi-M-norm, then $\|\ \|$ has the (weakened) M-property for *disjoint* vectors. Therefore, one strategy to find sufficient conditions for an quasi-M-norm to be equivalent to an M-norm, is to find conditions enabling to extend the weakened M-property for disjoint vectors to arbitrary positive vectors.

To do that, we want to approximate arbitrary positive vectors relatively uniformly by positive disjoint vectors. If E is weak-Freudenthal such an approximation is possible. However, if E is not weak-Freudenthal, we can still extend E to a Riesz space F that is (such as its lateral completion), and hope that we can extend the M-quasi-norm to F. The latter requires some sort of Fatou-continuity of $\|\ \|$, for which we introduce the following notion:

a set $U \subset E$ is called half-disjoint if $U = U_1 \cup U_2$, where both U_1 and U_2 are disjoint systems in E.

Lemma 2.42

Let E be a Riesz space and let $\|\ \| : E \to [0,\infty)$ be separating, absolutely homogeneous and Riesz.

(i) If $\|\ \|$ has the weakened M-property for disjoint elements, and if E is weak-Freudenthal, then $\|\ \|$ is a Riesz quasi-norm having the weakened M-property (and hence $\|\ \|$ is equivalent to an M-norm on E).

(ii) If $\|\ \|$ has the half-disjoint weak-Fatou property i.e.
there is a $C \in (0,\infty)$ such that

$$\left.\begin{array}{l} U \text{ half-disjoint in } E^+ \\ \sup U \text{ exists in } E^+ \end{array}\right\} \Rightarrow \|\sup U\| \le C \sup_{u\in U} \|u\| \qquad (*)$$

then $\|\ \|$ can be extended to the conditional lateral completion E_c^λ (p. 24) such that its extension has the weakened M-property for disjoint elements. By (i) the extension of $\|\ \|$ is then equivalent to an M-norm on E_c^λ.

In similar vein, we obtain by extension to the Dedekind completion E^δ:

(iii) If $\|\ \|$ is a weak-Fatou quasi-M-norm, then $\|\ \|$ is equivalent to an M-norm.

Proof

(i) Let E be weak-Freudenthal. Then lemma 0.66 is at our disposal:

Let $n \in \mathbb{N}$, $a_1,\dots,a_n$ in E^+, and $\varepsilon > 0$. Then there exist disjoint e_j $(1 \le j \le m)$ in E^+ and α_{ij} $(1 \le i \le n,\ 1 \le j \le m)$ in $[0,\infty)$ such that

$$\bigvee_j(\alpha_{ij} - \varepsilon)^+ e_j \le a_i \le \bigvee_j \alpha_{ij} e_j \qquad (1 \le i \le n)\,,$$

where

$$e := \textstyle\sum_1^n a_i = \sum_{j=1}^m e_j = \bigvee_{j=1}^m e_j\,.$$

(&)

Suppose now that $\|\ \| : E \to [0,\infty)$ is absolutely homogeneous, Riesz (so that in particular for $u_1,\dots,u_k$ in E^+ : $\vee_1^k \|u_i\| \le \| \vee_1^k u_i\|$) and that it has the weakened M-property for disjoint vectors i.e. there is a $K > 0$ such that

$$\|\bigvee_{j=1}^{N} u_j\| \le K \bigvee_{j=1}^{N} \|u_j\| \qquad \text{for all } u_1,\dots,u_N \text{ disjoint in } E^+ .$$

Then we have in the situation of (&):

$$\begin{aligned} \|a_i\| &\le \|\bigvee_j \alpha_{ij} e_j\| \le K \bigvee_j \alpha_{ij} \|e_j\| , \\ \|a_i\| &\ge \|\bigvee_j (\alpha_{ij} - \varepsilon)^+ e_j\| \ge \bigvee_j (\alpha_{ij} - \varepsilon)^+ \|e_j\| \\ &\ge \bigvee_j \alpha_{ij} \|e_j\| - \varepsilon \|\textstyle\sum_1^n a_i\| . \end{aligned} \tag{\$}$$

Taking $n = 2$, $\varepsilon > 0$, and setting $a := a_1$, $b := a_2$, $\alpha_j := \alpha_{1j}$ and $\beta_j := \alpha_{2j}$ in (&) and §, we get $a + b \le \vee_j(\alpha_j + \beta_j)e_j$ which implies

$$\begin{aligned} \|a+b\| &\le \left\|\bigvee_j (\alpha_j + \beta_j) e_j\right\| \le K \left[\bigvee_j \alpha_j \|e_j\| + \bigvee_j \beta_j \|e_j\|\right] \\ &\underset{(\$)}{\le} K \left[\|a\| + \|b\| + 2\varepsilon \|a+b\| \right] . \end{aligned}$$

Letting $\varepsilon \downarrow 0$ then establishes the quasi-subadditivity.

Taking $n \in \mathbb{N}$, and $\varepsilon > 0$ arbitrary in (&) gives:

$$\bigvee_i a_i \le \bigvee_{i,j} \alpha_{ij} e_j = \bigvee_j (\vee_i \alpha_{ij}) e_j ,$$

which implies

$$\begin{aligned} \|\bigvee_{i=1}^n a_i\| &\le \|\bigvee_j (\bigvee_i \alpha_{ij})\, e_j\| \le K \bigvee_j \|\bigvee_i \alpha_{ij} e_j\| \\ &= K \bigvee_i \left(\bigvee_j \alpha_{ij} \|e_j\|\right) \underset{(\$)}{\le} K \bigvee_i \left(\|a_i\| + \varepsilon \|\textstyle\sum_1^n a_i\| \right) . \end{aligned}$$

Letting $\varepsilon \downarrow 0$ then establishes the weakened M-property.

(ii) Since E^+ majorizes $(E_c^\lambda)^+$ we can define for $x^\lambda \in E_c^\lambda$:

$$\|x^\lambda\|^\lambda := \sup\{ \|u\| \ : \ u \in E , 0 \le u \le |x^\lambda| \} \in [0,\infty) .$$

Then $\|\,\|^\lambda : E_c^\lambda \to [0,\infty)$ is absolutely homogeneous and Riesz.
Suppose that there is a $C \in (0,\infty)$ such that $\|\sup U\| \le C \sup_{u \in U} \|u\|$ whenever U is a half-disjoint system in E^+ having a supremum in E.
Let $u^\lambda \in (E_c^\lambda)^+$ and $v \in E$ with $0 \le v \le u^\lambda$.
By choosing a half-disjoint system U in E^+ such that $\sup U = u^\lambda$ (0.76), we get $\|v\| \le C \sup_{u \in U} \|u \wedge v\| \le C \sup_{u \in U} \|u\|$. As a consequence:

for all half-disjoint systems $U \subset E^+$ with $\sup U = u^\lambda$:

$$\sup_{u \in U} \|u\| \le \left\|u^\lambda\right\|^\lambda \le C \sup_{u \in U} \|u\| \tag{\#}$$

If $u_1^\lambda, \dots, u_n^\lambda \in (E_c^\lambda)^+$ are disjoint, we can choose $U_1, \dots, U_n$ half-disjoint in E^+ with $\sup U_i = u_i^\lambda$ (all i) and $U_i \perp U_j$. Then $\cup_1^n U_i$ is half-disjoint with supremum $\vee_1^n u_i^\lambda$. Together with (#) that implies that $\|\,\|^\lambda$ has the weakened M-property for disjoint vectors. □

Recalling that $c_0(X)$, for a discrete set X, is essentially the only type of M-space that has an order-continuous norm (see [LiTz2, lemma 1.b.10, p. 19]), theorem 2.34 (p. 89), and the lemma above imply

Corollary 2.43
Let $(E, |\!|\!|\ |\!|\!|)$ be a quasi-Banach lattice such that all orthonormal sequences are equivalent.
If $|\!|\!|\ |\!|\!|$ has the weak-Fatou property, then E is topologically Riesz isomorphic to an L^p-space for some $p \in (0, \infty)$ or to a weak-Fatou M-space.
In particular, E is topologically Riesz isomorphic to an L^p-space ($p \in (0, \infty)$ or to a $c_0(X)$ for some discrete set X, if and only if $|\!|\!|\ |\!|\!|$ is order-continuous.

A second approach to find sufficient conditions on a quasi-M-space to be topologically Riesz isomorphic to an M-space, exploits the topological completeness of such spaces. By lemma 2.25 ii (p. 84) we have: if $(E, |\!|\!|\ |\!|\!|)$ is an quasi-Banach lattice and $\|\ \|$ an M-semi-norm on E, then $\|\ \|$ and $|\!|\!|\ |\!|\!|$ are equivalent if and only if they are equivalent on every principal ideal of E. Taking for $\|\ \|$ the M-semi-norm $|\!|\!|\ |\!|\!|_\vee$ of lemma 2.40 we obtain:

Theorem 2.44
Let $(E, |\!|\!|\ |\!|\!|)$ be a quasi-Banach lattice. Let $|\!|\!|\ |\!|\!|_\vee$ be the greatest M-semi-norm below $|\!|\!|\ |\!|\!|$ (see lemma 2.40, p. 97).
Take $e \in E^+$, let D be the principal ideal generated by e, and let

$$\|f\|_e := \inf\{r \in (0, \infty) : |f| \le re\} \quad (f \in D),$$

be the norm induced by e.

Suppose that D is $|\!|\!|\ |\!|\!|$-closed or Riesz isomorphic to a $C(X)$ with X metrizable. Then $|\!|\!|\ |\!|\!|\,|_D$, $|\!|\!|\ |\!|\!|_\vee\,|_D$, and $\|\ \|_e$ are equivalent.

In particular, by 2.25.ii, if every principal ideal of E is $|\!|\!|\ |\!|\!|$-closed or Riesz isomorphic to a $C(X)$ with X metrizable, then $|\!|\!|\ |\!|\!|$ and $|\!|\!|\ |\!|\!|_\vee$ are equivalent i.e. $(E, |\!|\!|\ |\!|\!|)$ is topologically Riesz isomorphic to an M-space.

Proof
Using Yoshida's theorem we identify D with $C(X)$ and e with $\mathbb{1} \in C(X)$. This way we have

$$\left.\begin{aligned} \|f\|_e &= \|f\|_\infty \\ |\!|\!|f|\!|\!|_\vee &= \|f\phi\|_\infty \end{aligned}\right\} \quad (f \in D),$$

where

$$\phi(x) := \inf\{\, |\!|\!|e|\!|\!| \,:\, e \in C(X), 0 \le e \le \mathbb{1}, e(x) = 1\} \quad (x \in X)$$

as in lemma 2.40.
To establish the equivalence of $|\!|\!|\ |\!|\!|$, $|\!|\!|\ |\!|\!|_\vee$, and $\|\ \|_e$ on D, we prove that

$$\text{for some } \varepsilon > 0 : \quad [\,\phi \ge \varepsilon\,] = X, \tag{\%}$$

for that implies

$$\varepsilon\|f\|_\infty \le \|f\phi\|_\infty = |\!|\!|f|\!|\!|_\vee \le |\!|\!|f|\!|\!| \le \|f\|_\infty\, |\!|\!|\mathbb{1}|\!|\!| \quad (f \in D).$$

Case that D is $|\!|\!|\ |\!|\!|$-closed
By applying the open-mapping theorem to the identity mapping $(E, |\!|\!|\ |\!|\!|) \to (E, \|\ \|_\infty)$, we see that there is an $\varepsilon > 0$ such that $\varepsilon\|\ \|_\infty \le |\!|\!|\ |\!|\!| \le |\!|\!|\mathbb{1}|\!|\!|\ \|\ \|_\infty$. Then $\phi \ge \varepsilon$ by its definition.

Case that X is metrizable

Suppose (%) does not hold. We will show that there exists a sequence $x_1, x_2, \dots$ in X converging to $x \in X$, and a $u \in C(X)$ such that $u(x_{2n}) = 1$, while $u(x_{2n+1}) = 0$ for all n.

Negating (%) means that $[\ \phi < \varepsilon\ ] \neq \emptyset$ for all $\varepsilon > 0$. As a result, the sets $[\ \phi < \varepsilon\ ]$, $\varepsilon > 0$, are not only open, but in addition infinite.

$\lhd$ Indeed, for each $\varepsilon > 0$

$$[\ \phi < \varepsilon\ ] = \bigcup_{0 \neq u \in C(X)^+} [\ u/\ \lVert u \rVert\ > \varepsilon^{-1}\ ] \quad \text{is open.}$$

Further, if $U = \{x_1, \dots, x_n\}$ is a finite open set in X, then for each i:

$$\{x_i\} = U \cap X \setminus \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} \quad \text{is open,}$$

and $\phi(x_i) = \left\lVert \mathbb{1}_{\{x_i\}} \right\rVert > 0$. Thus, $\inf_U \phi > 0$ for open finite subsets U. $\rhd$

That, together with the sequential compactness of X, allows us to choose $x_1, x_2, \dots$ in X such that $x_i \neq x_j$ $(i \neq j)$, $\phi(x_i) < 2^{-i}$ (all i) and $x_i \to x$ for some $x \in X$ with $x_i \neq x$ (all i). Using that $x \neq x_i \to x$, and passing to a subsequence if necessary, we can inductively find open sets $U_1, U_2, \dots$ and $V_1, V_2, \dots$ such that for all i:

$$x \in V_i\,,\ \ x_i \in U_i\,,\ \ V_i \cap U_i = \emptyset\,,\ \ V_i \supset V_{i+1}\,,\ \text{ and }\ U_{i+1} \subset V_i\,.$$

In particular, all U_i are disjoint. Since $\phi(x_i) < 2^{-i}$ and $\lVert\ \rVert$ is Riesz, we can find $u_i \in C(X)^+$ such that

$$0 \le u_i \le \mathbb{1}\,,\ u(x_i) = 1\,,\ \lVert u_i \rVert\ < 2^{-i}\,,\ \text{and } [\ u_i > 0\ ] \subset U_i\,,$$

i.e. $(u_i)_i$ is a disjoint sequence majorized by $\mathbb{1} \in C(X)$. By the topological completeness, $u := \sum_n u_{2n}$ exists in E, and since the lattice operations are continuous, u is majorized by $\mathbb{1} \in C(X)$ too. Thus, in fact, $u \in C(X)$. For all n we have $v_{2n+1} := \mathbb{1} - u_{2n+1} \in C(X)^+$ with $0 \le v_{2n+1} \le \mathbb{1}$, $v_{2n+1} = 1$ outside U_{2n+1}, and $v_{2n+1}(x_{2n+1}) = 0$. Then $v_{2n+1} \ge u \ge u_{2n}$, which implies that $u(x_{2n}) = 1$ and $u(x_{2n+1}) = 0$ for all n. The latter *contradicts* the continuity of u at x, and thereby establishes (%).

□

A third way to find sufficient conditions on an quasi-M-space to be topologically Riesz isomorphic to an M-space, is to use the dual of the quasi-M-space. However, the dual could be trivial, or even if it is separating, we still face the question how to translate information about the dual back to the quasi-M-space. Not subtle, but effective is requiring local convexity. No new result is obtained this way (in fact we only reformulate one of the characterizations of an M-space as a Banach-lattice [M-N, Theorem 2.1.12.iv, p. 59]), but the corollary (2.46) is worth mentioning.

Lemma 2.45

Let $(E,\ \lVert\ \rVert\ )$ be a quasi-M-space.

Suppose that the topology of E is normable i.e. there is a norm $\|\ \|$ on E that is equivalent to $\lVert\ \rVert$.

Then E is topologically Riesz isomorphic to an M-space.

Proof

Indeed, by 0.108, we may assume that $\|\ \|$ is a Riesz norm, which has - by its equivalence with $\lVert\ \rVert$ - the weakened M-property for disjoint elements. In [M-N, theorem 2.1.12(iv) ⇒ (i), p. 61] we see that this implies that for the dual-norm $\| \|'$ there is a $C > 0$ such that for all $\phi_1, \ldots, \phi_n$ in E'^+ disjoint:

$$\|\phi_1 + \ldots + \phi_n\|' \geq C(\|\phi_1\|' + \ldots + \|\phi_n\|')$$

i.e. all orthonormal sequences in $(E', \| \|')$ are equivalent with the standard basis of ℓ^1, so $(E', \| \|')$ is topologically isomorphic to an L^1-space, and therefore the second dual $(E'', \| \|'')$ is topologically Riesz isomorphic to an M-space (0.117). By standard Banach space theory E embeds isometrically in its bidual, and that finishes the proof. □

Combining theorem 2.34 and the lemma above yields the following.

Corollary 2.46
Let E be an infinite dimensional Banach lattice. Then E is topologically Riesz isomorphic to an L^p-space or an M-space if and only if every two orthonormal sequences in E are equivalent.

For completeness, we must also mention the following consequence of the open mapping theorem (cf. lemma 2.44):

Lemma 2.47
Any quasi-Banach lattice that has a strong order unit, is topologically Riesz isomorphic to an M-space.

Open problem 2.48
Roughly speaking, in two cases we have been able to prove that a quasi-M-space $(E, \rrbracket \llbracket\ )$ is topologically Riesz isomorphic to an M-space. Both cases address to the local structure of E: if the principal ideals of E are either "small" (Riesz isomorphic to a $C(X)$ with X metrizable) or contain sufficiently many disjoint elements (Riesz isomorphic to a $C(X)$ with X zero-dimensional), then $(E, \rrbracket \llbracket\ )$ is topologically Riesz isomorphic to an M-space. Of course, there exists a $C(X)$ which has neither properties such as $BC(\beta[0,\infty)/[0,\infty)\,)$.
Less far away from imagination lies the case of a principal ideal that is Riesz isomorphic to $BC(\mathbb{R}) \simeq C(\beta\mathbb{R})$. In that case, we have that ϕ (of lemma 2.40) is bounded away from zero on each bounded interval $I \subset \mathbb{R}$. In particular, we see that $\phi(x) > 0$ $(x \in \mathbb{R})$ which implies that $\rrbracket \llbracket_\vee$ is a norm (cf. 2.44). But the question remains whether $\rrbracket \llbracket$ is equivalent to $\rrbracket \llbracket_\vee$ on $BC(\mathbb{R})$.

Remarks 2.49
Ad 2.39 In the context of normed Riesz spaces there is no difference between the M-property and ∞-additivity: an ∞-additive Riesz norm is an M-norm (see [Be]).

Ad 2.40 The approach to define $\rrbracket u \llbracket_\vee$ could be described as refining "coverings of u" (if $\vee_\iota u_\iota = u$, the graphs of u_ι "cover" the graph of u). Another approach would be to refine disjoint "partitions" of u.

Call a *finite* collection of bands $D = \{B_1, \ldots, B_n\}$ a *partition* of (a Riesz space E) if $\vee_1^n B_\iota = E$ and $B_\iota \wedge B_\jmath = 0$ in the Boolean algebra of bands (i.e. $\cup_1^n B_\iota$ is order dense in E and the B_ι are mutually disjoint). Let $(E, \rrbracket \llbracket\ )$ be a quasi-normed Riesz space. For a partition $D = \{B_1, \ldots, B_n\}$ of E, and $x \in E$ we define

$$\rrbracket x \llbracket_D := \sup\{ \textstyle\bigvee_1^n \rrbracket u_\iota \llbracket \ : \ 0 \leq u_\iota \leq |x|,\ u_\iota \in B_\iota \text{ all } i \}\,.$$

The collection $\mathcal{D}$ of partitions of E can be directed by refinement: D' is a refinement of D ($D' \prec D$) if each $B \in D$ is generated by a subset of D' (i.e. there are $B'_1, \dots, B'_m$ in D' such that $\bigcup_1^m B'_j$ is order dense in B). If $D' \prec D$, $\|x\|'_D \leq \|x\|_{D'}$ and so we set

$$\|x\|_\perp := \lim_{D\in\mathcal{D}} \|x\|_D = \inf_{D\in\mathcal{D}} \|x\|_D \quad (x \in E)\ .$$

However, this way we do not obtain the greatest M-semi-norm below $\|\,\|$: if

$$E = C[0,1]\,, \quad \|f\| = \|f \cdot (\textstyle\sum_{n=1}^\infty 2^{-n}\mathbb{1}_{\{q_n\}})\|_\infty\ ,$$

where $\{q_1, q_2, \dots\}$ is a denumeration of $\mathbb{Q}\cap[0,1]$, then $\|f\|_\perp = 0$. For if D_n contains only bands whose elements have support outside $\{q_1, \dots, q_n\}$, then we see that $\|f\|_{D_n} \leq 2^{-n}\,\|f\|_\infty$ for all $f \in E$.

Ad 2.40 (4) The proof given for (4) of theorem 2.40 is an adaption of the way one proves that every M-norm $\|\,\|$ on $C(X)$ is of the form $\|f\| = \|f\phi\|_\infty$ where $\phi(x) := \inf\{\,\|u\| : 0 \leq u \in C(X)\,,\ u(x) = 1\}$.

Ad 2.42. Strictly speaking Fatou properties are about upwards directed sets. In the definition of half-disjoint (weak-)Fatou it is not the half-disjoint system that is upwards directed, but its collection of finite partial sums. The application we have in mind with 2.42 is clear: $\|\,\|$ is a quasi-M-norm with a weak-Fatou property for upwards directed sets that are induced by half-disjoint systems. However, even for this application we should not take the term too literally, because it comes down to a weak-Fatou property for upwards directed sets that are induced by disjoint systems combined with an M-property for suprema of such systems.

Two variants of (ii) are: *if E is almost σ-Dedekind complete (*or: has the countable sup property*) and $\|\,\|$ is weak σ-Fatou (*or: half-disjoint weak-σ-Fatou*), then $\|\,\|$ can be extended to the σ-Dedekind completion of E (*the conditional σ-lateral completion respectively*), and the extension has the M-property for disjoint vectors.*
In both cases the condition on E is meant to guarantee that each vector in the completion under consideration is a supremum of a *countable* upwards-directed or half-disjoint system respectively (0.78, 0.70). The proofs then run analogously to that presented in 2.42 (ii).

Ad 2.44. The essence of the proof of 2.44 is not to find a convergent sequence, but to find two countable subsets that have a common accumulation point. For that, weaker conditions than metrizability could be considered.

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Samenvatting

Titel: Karakteriseringen van L^p-ruimten met $p \in (0, \infty)$

In dit proefschrift worden karakteriseringen van L^p-ruimten met $p \in (0, \infty)$ bestudeerd. Voor het geval dat $p \in [1, \infty)$ bestaat er de inmiddels klassieke karakterisering van L^p-ruimten binnen de klasse van Banachtralies, bewezen door Kakutani en Bohnenblust. Echter, wanneer $p \in (0, 1)$, zijn L^p-ruimten geen Banachruimten. De grote manco die daardoor optreedt, is dat de gebruikelijke convexiteitsargumenten (bijvoorbeeld het gebruik van de duale) hun geldigheid verliezen. Wat overblijft, is gebruik te maken van bestaande theorie m.b.t. quasi-Banachtralies en die, waar nodig en mogelijk, uit te breiden.
Het proefschrift begint met een grondig en uitgebreid overzicht van de relevante theorie en reeds verkregen resultaten in het vakgebied. Daarna wordt in hoofdstuk 1 de Kakutani-Bohnenblust karakterisering uitgebreid tot het geval $p \in (0, \infty)$. In hoofdstuk 2 worden twee karakteriseringen van Tzafriri gegeneraliseerd. Ando's karakterisering d.m.v. positieve projecties blijkt echter niet uitgebreid te kunnen worden tot het geval dat $p \in (0, 1)$. De reden hiervoor wordt geanalyseerd in sectie 1.4. Een bijproduct van de analyse is een karakterisering van de bandprojecties binnen de klasse van contractieve projecties in L^p met $p \in (0, 1)$.
Tot slot wordt een nieuwe klasse ruimten, zogeheten quasi-M-ruimten, ingevoerd. Quasi-M-ruimten zijn een generalisatie van M-ruimten en komen op een natuurlijke manier naar voren bij het generaliseren van Tzafriri's karakterisering van L^p-ruimten en M-ruimten. Enkele voorwaarden waaronder quasi-M-ruimten in feite M-ruimten zijn, worden bestudeerd. De vraag of quasi-M-ruimten altijd M-ruimten zijn, blijft echter open.

Curriculum Vitae

Steven Teerenstra was born in Leiden and received his secondary education from the Stedelijk Gymnasium in Leeuwarden. From 1990-1996 he studied Mathematics (MSc cum laude, 1996) at the Katholieke Universiteit Nijmegen in parallel with Theoretical Physics from 1992-1997. His Masters thesis "Convoluties met Dirichlet- en Féjèrkern-achtige functies" was awarded a Universitaire Studieprijs in 1997. From 1996-2002 he was a part-time PhD-student in Mathematics under supervision of prof. A.C.M. van Rooij, together with whom he organized in 2001 the second international conference on Positivity and its Applications. During his study he had several sidelines in the Department of Anesthesiology and Intensive Care of the Universitair Medisch Centrum St. Radboud. Since 2003, he is added to the Department of Epidemiology and Biostatistics for working on biostatistical research projects. Outside study and work he is active for the Studentenkerk Nijmegen as sexton and editor of "Proviand" from 1999 onwards.